Continuum Mechanics

Chapter 9
Fluid Mechanics

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Introduction

Hydrostatic Pressure

There exist experimental evidence that the stress state of a *fluid at rest* is *hydrostatic* and it is characterized by a *spherical stress tensor* given by,

$$\boldsymbol{\sigma} = -p_0 \mathbf{1}$$

where $p_0 > 0$ is a positive scalar-valued quantity denoted as *hydrostatic pressure*.

The traction vector for a fluid at rest, at a given spatial point, is the same on any arbitrary plane with unit normal **n**, and is given by a compression state along the unit normal,

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = -p_0 \mathbf{1}\mathbf{n} = -p_0 \mathbf{n}$$

Introduction

Mean Pressure

The *mean pressure,* denoted as $\bar{p}$, is a scalar-valued quantity defined as minus the mean stress,

$$\bar{p} := -\sigma_m = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$$

For a *fluid at rest*, the *mean pressure* is equal to the *hydrostatic pressure*,

$$\bar{p} := -\sigma_m = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = -\frac{1}{3}\operatorname{tr}\left(-p_0\mathbf{1}\right) = p_0$$

Introduction

Thermodynamic Pressure

The *thermodynamic pressure,* denoted as p, is a scalar-valued quantity, that satisfies the following *kinetic state equation*,

$$\boxed{F(\rho, p, \theta) = 0}$$

For a *fluid at rest*, the *hydrostatic pressure* satisfies the kinetic state equation and, therefore, it is equal to the *thermodynamic pressure* yielding,

$$p = \overline{p} = p_0$$

For a fluid in motion the three pressures would be different,

$$p \neq \overline{p}, \quad p \neq p_0, \quad \overline{p} \neq p_0$$

Introduction

Barotropic Fluid

A fluid is said to be *barotropic* if the kinetic state equation does not depends on the temperature.

The *kinetic state equation* for a *barotropic fluid* may be written as,

$$F(\rho, p) = 0 \quad \Rightarrow \quad \rho = \rho(p)$$

A particular case of *barotropic fluid* is the *incompressible fluid*.

The *kinetic state equation* for an incompressible fluid may be written as,

$$F(\rho) = 0 \quad \Rightarrow \quad \rho = \rho_0$$

Constitutive Equations

Constitutive Equation for Stokes Fluids

The *constitutive equation* for a *Stokes fluid* may be written as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \mathbf{f}(\mathbf{d}, p, \theta), \quad \sigma_{ab} = -p\delta_{ab} + f_{ab}(\mathbf{d}, p, \theta)$$

- Ideal fluid:

$$\mathbf{f}(\mathbf{d}, p, \theta) = \mathbf{0}$$

- Newtonian fluid:

$$\mathbf{f}(\mathbf{d}, p, \theta) = K_0(p, \theta) I_1(\mathbf{d})\mathbf{1} + K_1(p, \theta)\mathbf{d}$$

Constitutive Equations

Constitutive Equation for Stokes Fluids

- Quasi-Newtonian fluid:

$$\begin{aligned}\mathbf{f}\left(\mathbf{d}, p, \theta\right) = & K_0\left(I_1\left(\mathbf{d}\right), I_2\left(\mathbf{d}\right), I_3\left(\mathbf{d}\right), p, \theta\right)\mathbf{1} \\ & + K_1\left(I_1\left(\mathbf{d}\right), I_2\left(\mathbf{d}\right), I_3\left(\mathbf{d}\right), p, \theta\right)\mathbf{d}\end{aligned}$$

- Reiner-Rivlin fluid:

$$\begin{aligned}\mathbf{f}\left(\mathbf{d}, p, \theta\right) = & K_0\left(I_1\left(\mathbf{d}\right), I_2\left(\mathbf{d}\right), I_3\left(\mathbf{d}\right), p, \theta\right)\mathbf{1} \\ & + K_1\left(I_1\left(\mathbf{d}\right), I_2\left(\mathbf{d}\right), I_3\left(\mathbf{d}\right), p, \theta\right)\mathbf{d} \\ & + K_2\left(I_1\left(\mathbf{d}\right), I_2\left(\mathbf{d}\right), I_3\left(\mathbf{d}\right), p, \theta\right)\mathbf{dd}\end{aligned}$$

Constitutive Equations

Constitutive Equation for Isotropic Newtonian Fluids

The *constitutive equation* for an isotropic *Newtonian fluid* may be written as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda(p,\theta)(\mathrm{tr}\,\mathbf{d})\mathbf{1} + 2\mu(p,\theta)\mathbf{d}$$

where $\lambda(p,\theta), \mu(p,\theta) \geq 0$ are two scalar-valued functions denoted as *dynamic viscosities.*

Constitutive Equations

Dynamic Viscosities for Isotropic Newtonian Fluids

The *dynamic viscosity* $\mu(p) \geq 0$ for an isotropic *Newtonian fluid* may be written as,

$$\mu(p) = \mu_0 \exp\left(\frac{p}{B}\right) \geq 0$$

where $\mu_0 \geq 0$ is a thermodynamic pressure-independent viscosity coefficient and B is a constant, usually bigger enough such that the dynamic viscosity coefficient $\mu = \mu(p) \geq 0$ can be considered as thermodynamic pressure-independent.

Constitutive Equations

Dynamic Viscosities for Isotropic Newtonian Fluids

The *dynamic viscosity* $\mu(\theta) \geq 0$ for an isotropic *Newtonian fluid* may be written as,

$$\mu(\theta) = \mu_0 \exp\left(\frac{Q}{R\theta}\right) \geq 0$$

where $\mu_0 \geq 0$ is a thermodynamic pressure-independent viscosity coefficient, Q is an activation energy and R is the universal constant of ideal gases.

Constitutive Equations

Dynamic Viscosities for Isotropic Newtonian Fluids

The *dynamic viscosity* $\mu(\theta) \geq 0$ for an isotropic *Newtonian fluid* may be written as,

$$\mu(\theta) = \mu_0 \exp\left(\frac{Q}{R\theta}\right) = \bar{\mu}_0 \exp\left(-\alpha\left(\theta - \theta_0\right)\right) \geq 0$$

where the following parameters have been introduced as,

$$\bar{\mu}_0 = \mu_0 \exp\left(\frac{Q}{R\theta_0}\right) \geq 0, \quad \alpha = -\frac{Q}{R\theta_0^2}$$

Constitutive Equations

Dynamic Viscosities for Quasi-Newtonian Fluids

Power law model. The *dynamic viscosity* $\mu(\theta) \geq 0$ for a *Quasi-Newtonian fluid* may be written as,

$$\mu\left(I_2(\mathbf{d})\right) = K_0\left(4I_2(\mathbf{d})\right)^{\frac{n-1}{2}}$$

where K_0 is the consistency parameter and n is the rate sensitivity coefficient.

Constitutive Equations

Dynamic Viscosities for Quasi-Newtonian Fluids

Carreau model. The *dynamic viscosity* $\mu(\theta) \geq 0$ for a *Quasi-Newtonian fluid* may be written as,

$$\mu\left(I_2(\mathbf{d})\right) = \mu_0 \left(1 + 4\lambda^2 I_2(\mathbf{d})\right)^{\frac{n-1}{2}}, \quad 0 < n < 1$$

where μ_0 is the constant dynamic viscosity parameter, λ is a model parameter and n is the rate sensitivity coefficient.

Governing Equations

Governing Equations

- **Conservation of mass.** Mass continuity equation

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy first motion equation

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Balance of energy**

1 $$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \operatorname{div} \mathbf{q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho \theta \dot{\eta} - \rho r + \operatorname{div} \mathbf{q} \geq 0, \quad \mathcal{D}_{con} := -\mathbf{q} \cdot \operatorname{grad} \theta \geq 0$$

1 1

Governing Equations

Constitutive Equations

- **Thermo-mechanical constitutive equation for the stresses**

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \mathbf{f}\left(\mathbf{d}, p, \theta\right)$$ 1

- **Thermo-mechanical constitutive equation for the entropy**

1 $$\eta = \eta\left(\mathbf{d}, p, \theta\right)$$

- **Thermal constitutive equation**. Fourier law

3 $$\mathbf{q} = \mathbf{q}\left(\mathbf{v}, \theta\right) = -\mathbf{k}\left(\mathbf{v}, \theta\right) \operatorname{grad} \theta$$

- **Caloric state equation**

1 $$e = e\left(\rho, \theta\right)$$

- **Kinetic state equation**

1 $$\rho = \rho\left(p, \theta\right)$$

Governing Equations

Mechanical Problem

- **Conservation of mass.** Mass continuity equation

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy first motion equation

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 9

- **Balance of angular momentum.** Symmetry of Cauchy stress

3 $$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

- **Mechanical constitutive equation** (temperature independent)

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \mathbf{f}(\mathbf{d}, p)$$ 1

- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho(p)$$

Governing Equations

Mechanical Problem

- **Conservation of mass.** Mass continuity equation

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

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- **Mechanical constitutive equation**

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- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho\left(p\right)$$

Governing Equations

Mechanical Problem

- **Conservation of mass.** Mass continuity equation

1 $$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$$ 1 3

- **Balance of linear momentum.** Cauchy first motion equation

3 $$-\operatorname{grad} p + \operatorname{div} \mathbf{f}(\mathbf{d}, p) + \rho \mathbf{b} = \rho \dot{\mathbf{v}}$$ 1

- **Kinetic state equation for a barotropic fluid**

1 $$\rho = \rho(p)$$

Governing Equations

Incompressible Mechanical Problem

- **Conservation of mass.** Mass continuity equation

1 $$\operatorname{div} \mathbf{v} = 0$$ 3

- **Balance of linear momentum.** Cauchy first motion equation

3 $$\operatorname{div} \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \rho_0 \dot{\mathbf{v}}$$ 6

- **Mechanical constitutive equation**

6 $$\boldsymbol{\sigma} = -p\mathbf{1} + \mathbf{f}(\mathbf{d}, p)$$ 1

Governing Equations

Incompressible Mechanical Problem

- **Conservation of mass.** Mass continuity equation

1 $$\operatorname{div}\mathbf{v} = 0$$ 3

- **Balance of linear momentum.** Cauchy first motion equation

3 $$-\operatorname{grad} p + \operatorname{div}\mathbf{f}\left(\mathbf{d}, p\right) + \rho_0\,\mathbf{b} = \rho_0\,\dot{\mathbf{v}}$$ 1

Governing Equations

Thermal Problem

- **Balance of energy**

1 $$\rho\,\dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho\, r - \operatorname{div}\mathbf{q}$$ 1 3

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho\,\theta\,\dot{\eta} - \rho\, r + \operatorname{div}\mathbf{q} \geq 0, \quad \mathcal{D}_{con} := -\mathbf{q}\cdot\operatorname{grad}\theta \geq 0$$

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- **Thermo-mechanical constitutive equation for the entropy**

1 $$\eta = \eta\left(\mathbf{d}, p, \theta\right)$$

- **Thermal constitutive equation**. Fourier law

3 $$\mathbf{q} = \mathbf{q}\left(\mathbf{v},\theta\right) = -\mathbf{k}\left(\mathbf{v},\theta\right)\operatorname{grad}\theta$$

- **Caloric state equation**

1 $$e = e\left(\rho,\theta\right)$$

Governing Equations

Thermal Problem

- **Balance of energy**

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- **Thermo-mechanical constitutive equation for the entropy**

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- **Caloric state equation**

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Governing Equations

Incompressible Thermal Problem

- **Balance of energy**

1 $$\rho_0 \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho_0 r + \operatorname{div}\left(\mathbf{k}\left(\mathbf{v},\theta\right) \operatorname{grad}\theta\right)$$ 1 1

- **Clausius-Planck and heat conduction inequalities**

$$\mathcal{D}_{int} := \rho_0 \theta \dot{\eta} - \rho_0 r - \operatorname{div}\left(\mathbf{k}\left(\mathbf{v},\theta\right) \operatorname{grad}\theta\right) \geq 0,$$

$$\mathcal{D}_{con} := \operatorname{grad}\theta \cdot \mathbf{k}\left(\mathbf{v},\theta\right) \operatorname{grad}\theta \geq 0$$ 1

- **Thermo-mechanical constitutive equation for the entropy**

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- **Caloric state equation**

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