Continuum Mechanics

Chapter 1
Introduction to Vectors and Tensors

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Introduction

Tensors

Continuum mechanics deals with *physical properties* of materials, either solids or fluids, which are independent of any particular coordinate system in which they are observed.

Those *physical properties* are mathematically represented by **tensors**.

Tensors are *mathematical objects* which have the required property of being *independent* of any particular coordinate system.

Introduction

Tensors

According to their *tensorial order*, tensors may be classified as:

- *Zero-order tensors*: scalars, i.e. density, temperature, pressure
- *First-order tensors*: vectors, i.e. velocity, acceleration, force
- *Second-order tensors*, i.e. stress, strain, strain rate
- *Third-order tensors*, i.e. piezoelectric tensor
- *Fourth-order tensors*, i.e. elastic constitutive tensor, elastoplastic constitutive tensor

First- and higher-order tensors may be expressed in terms of their *components* in a given coordinate system.

Introduction

Notation on Printed Documents

- *Zero-order tensors*: α, a, A
- *First-order tensors*: $\boldsymbol{\alpha}, \mathbf{a}, \mathbf{A}$
- *Second-order tensors*: $\boldsymbol{\alpha}, \mathbf{a}, \mathbf{A}$
- *Third-order tensors*: $\boldsymbol{\mathcal{A}}$
- *Fourth-order tensors*: $\mathbb{A}$

Notation on Hand-written Documents

- *Zero-order tensors*: α, a, A
- *First-order tensors*: $\underline{\alpha}, \underline{a}, \underline{A}$
- *Second-order tensors*: $\underline{\underline{\alpha}}, \underline{\underline{a}}, \underline{\underline{A}}$

Algebra of Vectors

Scalar

A physical quantity, completely described by a single real number, such as the *temperature*, *density* or *pressure*, is called a **scalar** and is designated by α, a, A.

A **scalar** may be viewed as a *zero-order tensor*.

Vector

A **vector** is a directed line element in space. A model for physical quantities having both direction and length, such as *velocity*, *acceleration* or *force*, is called a **vector** and is designated by.

$\boldsymbol{\alpha}, \mathbf{a}, \mathbf{A}$.

A **vector** may be viewed as a *first-order tensor*.

Algebra of Vectors

Sum of Vectors

The **sum** of vectors yields a new vector, based on the parallelo-gram law of addition.

The **sum** of vectors has the following **properties**,

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$$

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

$$\mathbf{u}+\mathbf{0}=\mathbf{u}$$

$$\mathbf{u}+(-\mathbf{u})=\mathbf{0}$$

where $\mathbf{0}$ denotes the unique **zero vector** with *unspecified* direction and zero length.

Algebra of Vectors

Scalar Multiplication

Let $\mathbf{u}$ be a vector and α be a scalar. The **scalar multiplication** $\alpha\mathbf{u}$ gives a new vector with the same direction as $\mathbf{u}$ if $\alpha > 0$ or with the opposite direction to $\mathbf{u}$ if $\alpha < 0$.

The **scalar multiplication** has the following **properties**,

$$(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$$

$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$$

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$$

Algebra of Vectors

Dot Product

The **dot** (or **scalar** or **inner**) **product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\cdot\mathbf{v}$, is a scalar given by,

$$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\left(\mathbf{u},\mathbf{v}\right),\quad 0\leq\theta\left(\mathbf{u},\mathbf{v}\right)\leq\pi$$

where $\|\mathbf{u}\|$ is the **norm** (or **length** or **magnitude**) of the vector $\mathbf{u}$, which is a non-negative real number defined as,

$$\|\mathbf{u}\|=\left(\mathbf{u}\cdot\mathbf{u}\right)^{1/2}\geq 0$$

and $\theta\left(\mathbf{u},\mathbf{v}\right)$ is the **angle** between the non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ when their origins coincide.

Algebra of Vectors

Dot Product

The **dot** (or **scalar** or **inner**) **product** of vectors has the following properties,

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$$

$$\mathbf{u}\cdot\mathbf{0}=0$$

$$\mathbf{u}\cdot(\alpha\mathbf{v}+\beta\mathbf{w})=\alpha(\mathbf{u}\cdot\mathbf{v})+\beta(\mathbf{u}\cdot\mathbf{w})$$

$$\mathbf{u}\cdot\mathbf{u}>0 \quad \Leftrightarrow \quad \mathbf{u}\neq\mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{u}=0 \quad \Leftrightarrow \quad \mathbf{u}=\mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{v}=0, \quad \mathbf{u}\neq\mathbf{0}, \mathbf{v}\neq\mathbf{0} \quad \Leftrightarrow \quad \mathbf{u}\perp\mathbf{v}$$

Algebra of Vectors

Unit Vectors

A vector $\mathbf{e}$ is called a **unit vector** if its norm (or lenght or magnitude) is equal to 1,

$$\|\mathbf{e}\| = 1$$

Orthogonal Vectors

A non-zero vector $\mathbf{u}$ is said to be **orthogonal** (or **perpendicular**) to a non-zero vector $\mathbf{v}$ if,

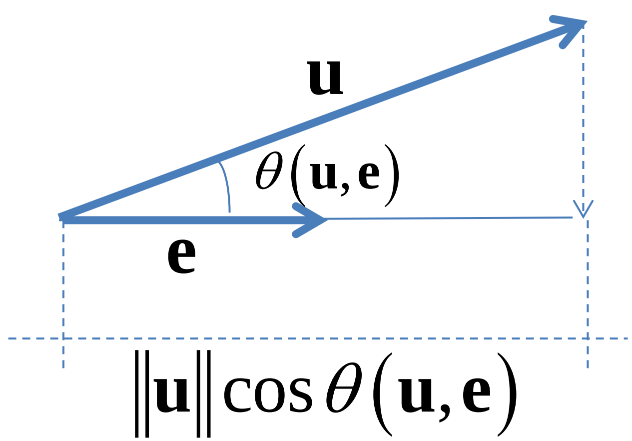
$$\mathbf{u} \cdot \mathbf{v} = 0, \quad \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0} \quad \Leftrightarrow \quad \theta(\mathbf{u}, \mathbf{v}) = \pi/2 \quad \Leftrightarrow \quad \mathbf{u} \perp \mathbf{v}$$

Algebra of Vectors

Projection

The **projection** of a vector $\mathbf{u}$ along a direction given by a unit vector $\mathbf{e}$ is a (positive or negative) scalar quantity defined as,

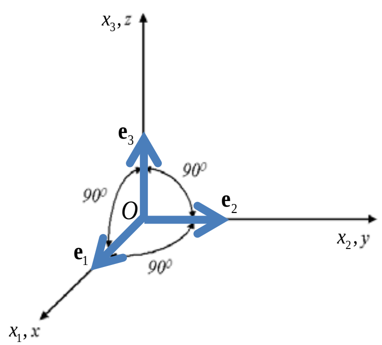
$$\mathbf{u}\cdot\mathbf{e} = \|\mathbf{u}\|\cos\theta(\mathbf{u},\mathbf{e}), \quad 0 \le \theta(\mathbf{u},\mathbf{e}) \le \pi$$



Algebra of Vectors

Cartesian Basis

Let us consider a **three-dimensional Euclidean space** with a fixed set of three **right-handed orthonormal basis vectors** $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, denoted as **Cartesian basis**, satisfying the following properties,



$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0$$

$$\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = \|\mathbf{e}_3\| = 1$$

A fixed set of three unit vectors which are mutually orthogonal form a so called **orthonormal basis vectors**, collectively denoted as $\{\mathbf{e}_i\}$, and define an **orthonormal system**.

Algebra of Vectors

Cartesian Components

Any **vector** $\mathbf{u}$ in the **three-dimensional Euclidean space** is represented **uniquely** by a linear combination of the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, i.e.

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

where the three real numbers u_1, u_2, u_3 are the uniquely determined **Cartesian components** of vector $\mathbf{u}$ along the given directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, respectively.

Using **matrix notation**, the Cartesian components of the vector $\mathbf{u}$ can be collected into a **vector of components** $[\mathbf{u}] \in \mathbb{R}^3$ given by,

$$[\mathbf{u}] = [u_1 \quad u_2 \quad u_3]^T$$

Algebra of Vectors

Index Notation

Using **index notation** the vector $\mathbf{u}$ can be written as,

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i$$

Leaving out the summation symbol, using the so called **Einstein notation**, the vector $\mathbf{u}$ can be written in short form as,

$$\mathbf{u} = u_i \mathbf{e}_i$$

where we have adopted the **summation convention** introduced by Einstein.

Algebra of Vectors

Index Notation

- The index that is summed over is said to be a **dummy** (or **summation**) **index**.
- The same **repeated index** can appear only **twice**.
- An index that is not summed over in a given term is called a **free** (or **live**) **index**.

$$v_i = a_i b_j c_j = a_i \left(b_1 c_1 + b_2 c_2 + b_3 c_3 \right)$$

$$v = a_i b_i c_j d_j = \left(a_1 b_1 + a_2 b_2 + a_3 b_3 \right)\left(c_1 d_1 + c_2 d_2 + c_3 d_3 \right)$$

$$v_i = a_i b_j c_j c_k d_k = a_i \left(b_1 c_1 + b_2 c_2 + b_3 c_3 \right)\left(c_1 d_1 + c_2 d_2 + c_3 d_3 \right)$$

Algebra of Vectors

Kronecker Delta

The **Kronecker delta** δ_{ij} is defined as,

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The **Kronecker delta** satisfies the following properties,

$$\delta_{ii} = 3, \quad \delta_{ij}\delta_{jk} = \delta_{ik}, \quad \delta_{ij}u_j = u_i$$

Note that the Kronecker delta plays the role of a **replacement operator**.

Algebra of Vectors

Dot Product

The **dot** (or **scalar** or **inner**) **product** of two orthonormal basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, denoted as $\mathbf{e}_i \cdot \mathbf{e}_j$, is a scalar quantity (taking values either 0 or 1) and can be conveniently written in terms of the **Kronecker delta** as,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

Algebra of Vectors

Components of a Vector

Taking the basis $\{\mathbf{e}_i\}$, the **projection** of a vector $\mathbf{u}$ onto the basis vector $\mathbf{e}_i$ yields the *i*-th **component** of $\mathbf{u}$,

$$\mathbf{u}\cdot\mathbf{e}_i = \left(u_j\mathbf{e}_j\right)\cdot\mathbf{e}_i = u_j\mathbf{e}_j\cdot\mathbf{e}_i = u_j\delta_{ji} = u_i$$

Algebra of Vectors

Dot Product

Taking the basis $\{\mathbf{e}_i\}$, the **dot** (or **scalar** or **inner**) **product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \cdot \mathbf{v}$, is a scalar quantity and using **index notation** can be written as,

$$\mathbf{u} \cdot \mathbf{v} = u_i \mathbf{e}_i \cdot v_j \mathbf{e}_j = u_i v_j \mathbf{e}_i \cdot \mathbf{e}_j = u_i v_j \delta_{ij} = u_i v_i$$
$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

Algebra of Vectors

Euclidean Norm

Taking the basis $\{\mathbf{e}_i\}$, the **Euclidean norm** (or **length** or **magnitude**) of a vector $\mathbf{u}$, denoted as $\|\mathbf{u}\|$, is a non-negative scalar quantity and, using **index notation**, can be written as,

$$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = \left(u_i\mathbf{e}_i \cdot u_j\mathbf{e}_j\right)^{1/2} = \left(u_i u_j \delta_{ij}\right)^{1/2} = \left(u_i u_i\right)^{1/2}$$

$$= \left(u_1^2 + u_2^2 + u_3^2\right)^{1/2}$$

Assignment 1.1

Assignment 1.1 [Classwork]

Write in index form the following expressions,

$$\mathbf{v} = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

$$\mathbf{v} = \mathbf{a}\,(\mathbf{b} \cdot \mathbf{c})$$

$$v = (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d})$$

$$\mathbf{v} = \mathbf{a}\,(\mathbf{b} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{d})$$

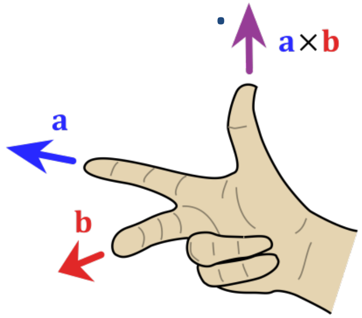
Algebra of Vectors

Cross Product

The **cross** (or **vector**) **product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u}\times\mathbf{v}$, is another vector which is perpendicular to the plane defined by the two vectors and its norm (or length) is given by the area of the parallelogram spanned by the two vectors,

$$\mathbf{u}\times\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta(\mathbf{u},\mathbf{v})\mathbf{e}, \quad 0 \le \theta(\mathbf{u},\mathbf{v}) \le \pi$$

where $\mathbf{e}$ is a unit vector perpendicular to the plane defined by the two vectors, $\mathbf{u}\cdot\mathbf{e}=0$, $\mathbf{v}\cdot\mathbf{e}=0$, in the direction given by the right-hand rule,

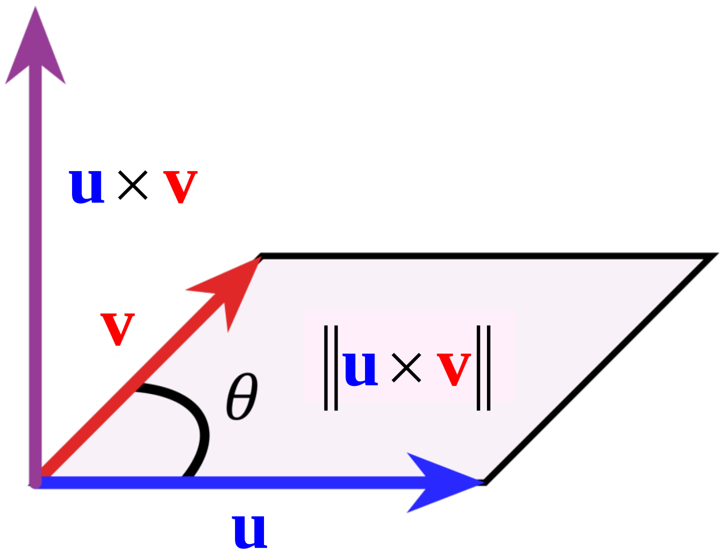


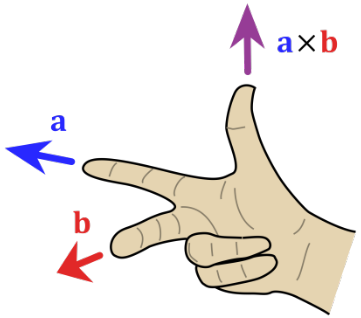
Algebra of Vectors

Norm of the Cross Product of two Vectors

The **norm** (or **magnitude** or **length**) of the **cross** (or **vector**) **product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\left\|\mathbf{u}\times\mathbf{v}\right\|$, measures the area spanned by the two vectors and is given by,

$$\left\|\mathbf{u}\times\mathbf{v}\right\| = \left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|\sin\theta\left(\mathbf{u},\mathbf{v}\right), \quad 0 \le \theta\left(\mathbf{u},\mathbf{v}\right) \le \pi$$



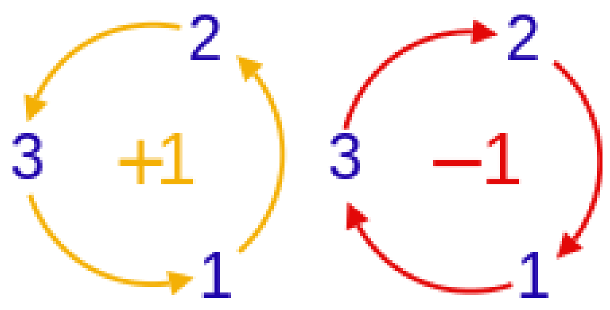


Algebra of Vectors

Permutation Symbol

The **permutation** (or **alternating** or **Levi-Civita**) **symbol** ε_{ijk} is defined as,

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{for even permutations of } (i, j, k) \text{, i.e. } 123,\ 231,\ 312 \\ -1 & \text{for odd permutations of } (i, j, k) \text{, i.e. } 213, 132,\ 321 \\ 0 & \text{if there is repeated index} \end{cases}$$



Algebra of Vectors

Permutation Symbol

The **permutation** (or **alternating** or **Levi-Civita**) **symbol** ε_{ijk} may be written in terms of the *Kronecker delta* and has the following properties,

$$\varepsilon_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad \varepsilon_{ijk}\varepsilon_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$$

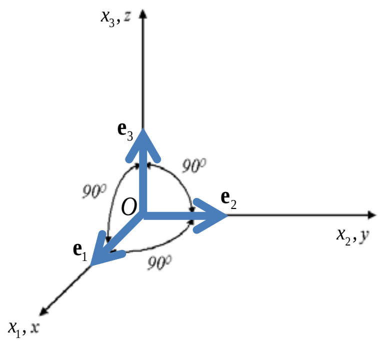
$$\varepsilon_{ijp}\varepsilon_{klp} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \quad \varepsilon_{ikl}\varepsilon_{jkl} = 2\delta_{ij}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 6$$

$$\varepsilon_{ijk} = \frac{(i-j)(j-k)(k-i)}{2}$$

Algebra of Vectors

Cross Product

The **cross** (or **vector**) **product** of two *right-handed orthonormal basis vectors* satisfies the following properties,



$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2,$$

$$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2,$$

$$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0}.$$

Taking the *right-handed orthonormal basis* $\{\mathbf{e}_i\}$, the **cross** (or **vector**) **product** of two basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, denoted as $\mathbf{e}_i \times \mathbf{e}_j$, can defined as,

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k$$

Algebra of Vectors

Cross Product

Taking the basis $\{\mathbf{e}_i\}$, the **cross** (or **vector**) **product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u}\times\mathbf{v}$, is another vector, perpendicular to the plane defined by the two vectors, and can be written in index form as,

$$\mathbf{u}\times\mathbf{v} = u_i\mathbf{e}_i \times v_j\mathbf{e}_j = u_i v_j \mathbf{e}_i \times \mathbf{e}_j = u_i v_j \varepsilon_{ijk}\mathbf{e}_k = \varepsilon_{ijk} u_i v_j \mathbf{e}_k$$
$$= \left(u_2 v_3 - u_3 v_2\right)\mathbf{e}_1 + \left(u_3 v_1 - u_1 v_3\right)\mathbf{e}_2 + \left(u_1 v_2 - u_2 v_1\right)\mathbf{e}_3$$
$$= \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Algebra of Vectors

Cross Product

The **cross** (or **vector**) **product** of vectors has the following properties,

$$\mathbf{u}\times\mathbf{v}=-\left(\mathbf{v}\times\mathbf{u}\right)$$

$$\mathbf{u}\times\mathbf{v}=\mathbf{0},\quad \mathbf{u}\neq\mathbf{0},\mathbf{v}\neq\mathbf{0}\quad\Leftrightarrow\quad \mathbf{u}\,\|\,\mathbf{v}$$

$$\left(\alpha\mathbf{u}\right)\times\mathbf{v}=\mathbf{u}\times\left(\alpha\mathbf{v}\right)=\alpha\left(\mathbf{u}\times\mathbf{v}\right)$$

$$\mathbf{u}\times\left(\mathbf{v}+\mathbf{w}\right)=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$$

Assignment 1.2

Assignment 1.2

Using the expression,

$$\left\|\mathbf{u}\times\mathbf{v}\right\|^2 = \left\|\mathbf{u}\right\|^2 \left\|\mathbf{v}\right\|^2 \sin^2\theta\left(\mathbf{u},\mathbf{v}\right), \quad 0 \le \theta\left(\mathbf{u},\mathbf{v}\right) \le \pi$$

obtain the following relationships between the permutation symbol and the Kronecker delta,

$$\varepsilon_{ijp}\varepsilon_{klp} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \quad \varepsilon_{ipq}\varepsilon_{jpq} = 2\delta_{ij}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 6$$

Algebra of Vectors

Box Product

The **box** (or **triple scalar**) **product** of three *right-handed orthonormal basis vectors* $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$, denoted as $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)$, is a scalar quantity (taking values either 1, -1 or 0) and can be conveniently written in terms of the permutation symbol as,

$$\varepsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k$$

Algebra of Vectors

Box Product

Taking the orthonormal basis $\{\mathbf{e}_i\}$, the **box** (or **triple scalar**) **product** of three vectors $\mathbf{u},\mathbf{v},\mathbf{w}$, denoted as $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, is a scalar quantity and using index notation can be written as,

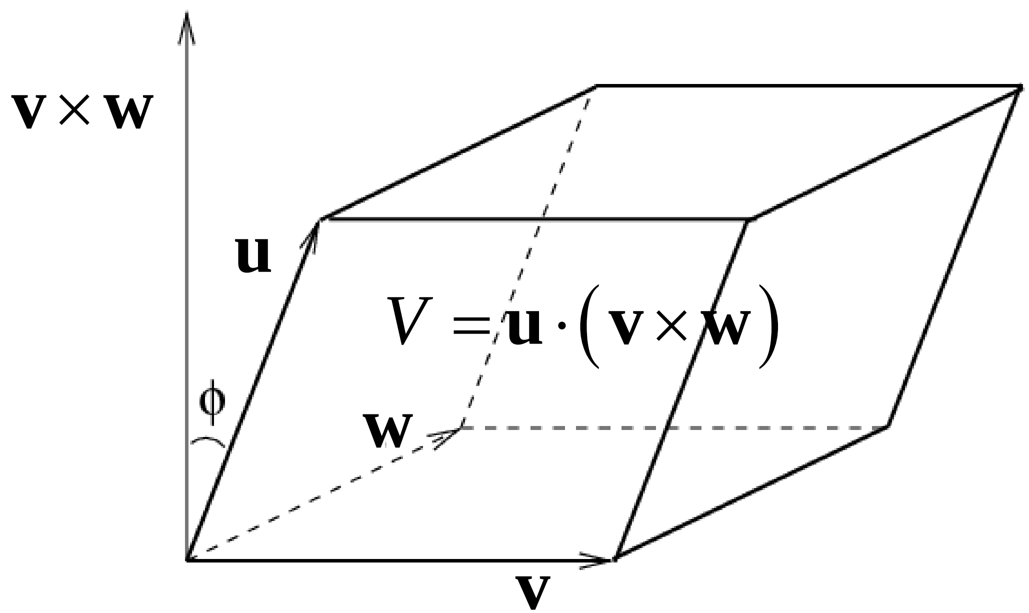
$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = u_i\mathbf{e}_i\cdot(v_j\mathbf{e}_j\times w_k\mathbf{e}_k) = u_i v_j w_k \mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k)$$
$$= \varepsilon_{ijk} u_i v_j w_k$$
$$= \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Algebra of Vectors

Box Product

The **box** (or **triple scalar**) **product** of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ represents the **volume** of the parallelepiped spanned by the three vectors forming a *right-handed* triad,

$$V = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$



Algebra of Vectors

Box Product

The **box** (or **triple scalar**) **product** has the following properties,

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=-\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})$$

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0$$

Algebra of Vectors

Triple Vector Product

Taking the orthonormal basis $\{\mathbf{e}_i\}$, the **triple vector product** of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, denoted as $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, is a vector and using index notation can be written as,

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \varepsilon_{ijk} u_i \left(\varepsilon_{pqj} v_p w_q \right) \mathbf{e}_k \\
&= \varepsilon_{kij} \varepsilon_{pqj} u_i v_p w_q \mathbf{e}_k \\
&= \left(\delta_{kp} \delta_{iq} - \delta_{kq} \delta_{ip} \right) u_i v_p w_q \mathbf{e}_k \\
&= u_i v_k w_i \mathbf{e}_k - u_i v_i w_k \mathbf{e}_k \\
&= (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}
\end{aligned}$$

Algebra of Vectors

Triple Vector Product

Taking the orthonormal basis $\{\mathbf{e}_i\}$, the **triple vector product** of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, denoted as $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$, is a vector and using index notation can be written as,

$$
\begin{aligned}
(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= \varepsilon_{ijk}\left(\varepsilon_{pqi}u_p v_q\right)w_j\mathbf{e}_k \\
&= \varepsilon_{jki}\varepsilon_{pqi}u_p v_q w_j\mathbf{e}_k \\
&= \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right)u_p v_q w_j\mathbf{e}_k \\
&= u_j w_j v_k\mathbf{e}_k - v_j w_j u_k\mathbf{e}_k \\
&= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}
\end{aligned}
$$

Algebra of Vectors

Triple Vector Product

The **triple vector product** has the following properties,

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\neq(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$$

Algebra of Tensors

Second-order Tensors

A **second-order tensor** $\mathbf{A}$ may be thought of as a **linear operator** that acts on a vector $\mathbf{u}$ generating a vector $\mathbf{v}$, defining a **linear transformation** that assigns a vector $\mathbf{v}$ to a vector $\mathbf{u}$,

$$\mathbf{v} = \mathbf{A}\mathbf{u}$$

The second-order **unit tensor**, denoted as $\mathbf{1}$, satisfies the following identity,

$$\mathbf{u} = \mathbf{1}\mathbf{u}$$

Algebra of Tensors

Second-order Tensors

As *linear operators* defining *linear transformations*, second-order tensors have the following **properties**,

$$\mathbf{A}(\alpha \mathbf{u} + \mathbf{v}) = \alpha \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}$$

$$(\alpha \mathbf{A})\mathbf{u} = \alpha (\mathbf{A}\mathbf{u})$$

$$(\mathbf{A} \pm \mathbf{B})\mathbf{u} = \mathbf{A}\mathbf{u} \pm \mathbf{B}\mathbf{u}$$

Algebra of Tensors

Tensor Product

The **tensor product** (or **dyad**) of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \otimes \mathbf{v}$, is a second-order tensor which **linearly transforms** a vector $\mathbf{w}$ into a vector with the direction of $\mathbf{u}$ following the rule,

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

A **dyadic** is a linear combination of dyads.

Algebra of Tensors

Tensor Product

The **tensor product** (or **dyad)** has the following linear properties,

$$(\mathbf{u}\otimes\mathbf{v})(\alpha\mathbf{w}+\mathbf{x})=\alpha(\mathbf{u}\otimes\mathbf{v})\mathbf{w}+(\mathbf{u}\otimes\mathbf{v})\mathbf{x}$$

$$(\alpha\mathbf{u}+\beta\mathbf{v})\otimes\mathbf{w}=\alpha\mathbf{u}\otimes\mathbf{w}+\beta\mathbf{v}\otimes\mathbf{w}$$

$$(\mathbf{u}\otimes\mathbf{v})\mathbf{w}=(\mathbf{v}\cdot\mathbf{w})\mathbf{u}=\mathbf{u}(\mathbf{v}\cdot\mathbf{w})$$

$$(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})=(\mathbf{v}\cdot\mathbf{w})\mathbf{u}\otimes\mathbf{x}=\mathbf{u}\otimes\mathbf{x}(\mathbf{v}\cdot\mathbf{w})$$

$$\mathbf{A}(\mathbf{u}\otimes\mathbf{v})=(\mathbf{A}\mathbf{u})\otimes\mathbf{v}$$

Generally, the tensor product (or dyad) is *not* commutative, i.e.

$$\mathbf{u}\otimes\mathbf{v}\neq\mathbf{v}\otimes\mathbf{u}$$

$$(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})\neq(\mathbf{w}\otimes\mathbf{x})(\mathbf{u}\otimes\mathbf{v})$$

Algebra of Tensors

Tensor Product

The **tensor product** (or **dyad**) of two orthonormal basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, denoted as $\mathbf{e}_i \otimes \mathbf{e}_j$, is a second-order tensor and can be thought as a *linear operator* such that,

$$\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_k = \left(\mathbf{e}_j \cdot \mathbf{e}_k\right)\mathbf{e}_i = \delta_{jk}\mathbf{e}_i$$

Algebra of Tensors

Second-order Tensors

Any **second-order tensor** $\mathbf{A}$ may be represented by a linear combination of dyads formed by the Cartesian basis $\{\mathbf{e}_i\}$

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

where the nine real numbers, represented by A_{ij}, are the uniquely determined **Cartesian components** of the tensor $\mathbf{A}$ with respect to the dyads formed by the Cartesian basis $\{\mathbf{e}_i\}$, represented by $\mathbf{e}_i \otimes \mathbf{e}_j$, which constitute a basis second-order tensor for $\mathbf{A}$.

The second-order **unit tensor** may be written as,

$$\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$$

Algebra of Tensors

Cartesian Components

The ***ij*-th Cartesian component** of the second-order tensor $\mathbf{A}$, denoted as A_{ij}, can be written as,

$$\mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j = \mathbf{e}_i \cdot \left(A_{kl} \mathbf{e}_k \otimes \mathbf{e}_l \right) \mathbf{e}_j = \mathbf{e}_i \cdot \left(A_{kl} \delta_{lj} \mathbf{e}_k \right)$$

$$= \mathbf{e}_i \cdot \left(A_{kj} \mathbf{e}_k \right) = A_{kj} \mathbf{e}_i \cdot \mathbf{e}_k = A_{kj} \delta_{ik} = A_{ij}$$

The ***ij*-th Cartesian component** of the second-order unit tensor $\mathbf{1}$ is the Kronecker delta,

$$\mathbf{e}_i \cdot \mathbf{1}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

Algebra of Tensors

Matrix of Components

Using **matrix notation**, the Cartesian components of any **second-order tensor** $\mathbf{A}$ may be collected into a 3 x 3 **matrix of components**, denoted by $[\mathbf{A}]$, given by,

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Algebra of Tensors

Matrix of Components

Using **matrix notation**, the Cartesian components of the second-order unit tensor $\mathbf{1}$ may be collected into a 3 x 3 **matrix of components**, denoted as $[\mathbf{1}]$, given by,

$$\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$$

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algebra of Tensors

Second-order Tensors

The **tensor product** (or **dyad**) of the vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \otimes \mathbf{v}$, may be represented by a linear combination of dyads formed by the Cartesian basis $\{\mathbf{e}_i\}$

$$\mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$$

where the nine real numbers, represented by $u_i v_j$, are the uniquely determined **Cartesian components** of the tensor $\mathbf{u} \otimes \mathbf{v}$ with respect to the dyads formed by the Cartesian basis $\{\mathbf{e}_i\}$, represented by $\mathbf{e}_i \otimes \mathbf{e}_j$.

Algebra of Tensors

Matrix of Components

Using **matrix notation**, the Cartesian components of any **second-order tensor** $\mathbf{u} \otimes \mathbf{v}$ may be collected into a 3 x 3 **matrix of components**, denoted as $[\mathbf{u} \otimes \mathbf{v}]$, given by,

$$\mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$$

$$[\mathbf{u} \otimes \mathbf{v}] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

Algebra of Tensors

Second-order Tensors

Using **index notation**, the linear transformation $\mathbf{v} = \mathbf{A}\mathbf{u}$ can be written as,

$$\mathbf{v} = \mathbf{A}\mathbf{u}$$

$$v_i \mathbf{e}_i = \left(A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \right) \left(u_k \mathbf{e}_k \right) = A_{ij} u_k \left(\mathbf{e}_i \otimes \mathbf{e}_j \right) \mathbf{e}_k = A_{ij} u_k \delta_{jk} \mathbf{e}_i$$

$$v_i \mathbf{e}_i = A_{ij} u_j \mathbf{e}_i$$

yielding

$$\boxed{\mathbf{v} = \mathbf{A}\mathbf{u}, \quad v_i = A_{ij} u_j, \quad [\mathbf{v}] = [\mathbf{A}][\mathbf{u}]}$$

Algebra of Tensors

Positive Semi-definite Second-order Tensor

A **second-order tensor** $\mathbf{A}$ is said to be a positive semi-definite tensor if the following relation holds for any non-zero vector $\mathbf{u} \neq \mathbf{0}$

$$\mathbf{u} \cdot \mathbf{A}\mathbf{u} \geq 0 \quad \forall \mathbf{u} \neq \mathbf{0}$$

Positive Definite Second-order Tensor

A **second-order tensor** $\mathbf{A}$ is said to be a positive-definite tensor if the following relation holds for any non-zero vector $\mathbf{u} \neq \mathbf{0}$

$$\mathbf{u} \cdot \mathbf{A}\mathbf{u} > 0 \quad \forall \mathbf{u} \neq \mathbf{0}$$

Algebra of Tensors

Transpose of a Second-order Tensor

The unique **transpose** of a second-order tensor $\mathbf{A}$, denoted as $\mathbf{A}^T$, is governed by the identity,

$$\mathbf{v}\cdot\mathbf{A}^T\mathbf{u}=\mathbf{u}\cdot\mathbf{A}\mathbf{v} \quad \forall\mathbf{u},\mathbf{v}$$

$$v_i A_{ij}^T u_j = v_i A_{ji} u_j = u_j A_{ji} v_i \quad \forall u_j, v_i$$

The **Cartesian components** of the transpose of a second-order tensor $\mathbf{A}$ satisfy,

$$A_{ij}^T := \left(\mathbf{A}^T\right)_{ij} = \mathbf{e}_i\cdot\mathbf{A}^T\mathbf{e}_j = \mathbf{e}_j\cdot\mathbf{A}\mathbf{e}_i = A_{ji}$$

Using **index notation** the transpose of $\mathbf{A}$ may be written as,

$$\mathbf{A}^T = A_{ji}\mathbf{e}_i\otimes\mathbf{e}_j$$

Algebra of Tensors

Transpose of a Second-order Tensor

The **matrix of components** of the **transpose** of a second-order tensor $\mathbf{A}$ is equal to the transpose of the matrix of components of $\mathbf{A}$ and takes the form,

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\mathbf{A}^T = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\left[\mathbf{A}^T\right] = [\mathbf{A}]^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Algebra of Tensors

Symmetric Second-order Tensor

A second-order tensor $\mathbf{A}$ is said to be **symmetric** if the following relation holds,

$$\mathbf{A} = \mathbf{A}^T$$

Using **index notation**, the Cartesian components of a symmetric second-order tensor $\mathbf{A}$ satisfy,

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{A}^T, \quad A_{ij} = A_{ji}$$

Using **matrix notation**, the matrix of components of a symmetric second-order tensor $\mathbf{A}$ satisfies,

$$[\mathbf{A}] = \left[\mathbf{A}^T\right] = [\mathbf{A}]^T$$

Algebra of Tensors

Dot Product

The **dot product** of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{AB}$, is a second-order tensor such that,

$$(\mathbf{AB})\mathbf{u} = \mathbf{A}(\mathbf{Bu}) \quad \forall \mathbf{u}$$

Algebra of Tensors

Dot Product

The **components** of the dot product $\mathbf{AB}$ along an orthonormal basis $\{\mathbf{e}_i\}$ read,

$$C_{ij} = (\mathbf{AB})_{ij} = \mathbf{e}_i \cdot (\mathbf{AB})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}(\mathbf{B}\mathbf{e}_j)$$

$$= \mathbf{e}_i \cdot \mathbf{A}(B_{kj}\mathbf{e}_k) = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_k B_{kj}$$

$$= A_{ik} B_{kj}$$

$$\mathbf{C} = \mathbf{AB}, \quad C_{ij} = A_{ik} B_{kj}$$

Algebra of Tensors

Dot Product

The **dot product** of second-order tensors has the following properties,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$$

$$\mathbf{A}^2 = \mathbf{AA}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

Algebra of Tensors

Trace

The **trace** of a dyad $\mathbf{u} \otimes \mathbf{v}$, denoted as $\mathrm{tr}\left(\mathbf{u} \otimes \mathbf{v}\right)$, is a scalar quantity given by,

$$\mathrm{tr}\left(\mathbf{u} \otimes \mathbf{v}\right) = \mathbf{u} \cdot \mathbf{v} = u_i v_i$$

The **trace** of a second-order tensor $\mathbf{A}$, denoted as $\mathrm{tr}\,\mathbf{A}$, is a scalar quantity given by,

$$\begin{aligned} \mathrm{tr}\,\mathbf{A} &= \mathrm{tr}\left(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) = A_{ij}\,\mathrm{tr}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= A_{ij}\left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = A_{ij}\delta_{ij} \\ &= A_{ii} \\ &= A_{11} + A_{22} + A_{33} = \mathrm{tr}\left[\mathbf{A}\right] \end{aligned}$$

Algebra of Tensors

Trace

The **trace** of a second-order tensor has the following properties,

$$\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{A}^T$$

$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$$

$$\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$$

$$\operatorname{tr}(\alpha\mathbf{A}) = \alpha\operatorname{tr}(\mathbf{A})$$

Algebra of Tensors

Double Dot Product

The **double dot product** of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A}:\mathbf{B}$, is a scalar quantity defined as,

$$\begin{aligned}\mathbf{A}:\mathbf{B} &= \operatorname{tr}\left(\mathbf{A}^T\mathbf{B}\right) = \operatorname{tr}\left(\mathbf{B}^T\mathbf{A}\right) \\ &= \operatorname{tr}\left(\mathbf{A}\mathbf{B}^T\right) = \operatorname{tr}\left(\mathbf{B}\mathbf{A}^T\right) \\ &= \mathbf{B}:\mathbf{A}\end{aligned}$$

or using **index notation**,

$$\mathbf{A}:\mathbf{B} = A_{ij}B_{ij} = B_{ij}A_{ij} = \mathbf{B}:\mathbf{A}$$

Algebra of Tensors

Double Dot Product

The **double dot product** has the following properties,

$$\mathbf{A} : \mathbf{1} = \mathbf{1} : \mathbf{A} = \operatorname{tr} \mathbf{A}$$

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A}$$

$$\mathbf{A} : (\mathbf{B}\mathbf{C}) = \left(\mathbf{B}^T \mathbf{A}\right) : \mathbf{C} = \left(\mathbf{A}\mathbf{C}^T\right) : \mathbf{B}$$

$$\mathbf{A} : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{A}\mathbf{v} = (\mathbf{u} \otimes \mathbf{v}) : \mathbf{A}$$

$$(\mathbf{u} \otimes \mathbf{v}) : (\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x})$$

$$\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) : \left(\mathbf{e}_k \otimes \mathbf{e}_l\right) = \left(\mathbf{e}_i \cdot \mathbf{e}_k\right)\left(\mathbf{e}_j \cdot \mathbf{e}_l\right) = \delta_{ik}\delta_{jl}$$

Algebra of Tensors

Euclidean Norm

The **euclidean norm** of a second-order tensor $\mathbf{A}$ is a non-negative scalar quantity, denoted as $\|\mathbf{A}\|$, given by,

$$\|\mathbf{A}\| = (\mathbf{A} : \mathbf{A})^{1/2} = \left(A_{ij} A_{ij}\right)^{1/2} \geq 0$$

Algebra of Tensors

Determinant

The **determinant** of a second-order tensor $\mathbf{A}$ is a scalar quantity, denoted as $\det \mathbf{A}$, given by,

$$\det \mathbf{A} = \det[\mathbf{A}]$$

$$= \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} A_{pi} A_{qj} A_{rk}$$

Algebra of Tensors

Determinant

The determinant of a second-order tensor has the following properties,

$$\det \mathbf{A}^T = \det \mathbf{A}$$

$$\det(\alpha \mathbf{A}) = \alpha^3 \det \mathbf{A}$$

$$\det(\mathbf{A}\mathbf{B}) = \det \mathbf{A} \det \mathbf{B}$$

Algebra of Tensors

Singular Tensors

A second-order tensor $\mathbf{A}$ is said to be **singular** *if and only if* its determinant is equal to zero, i.e.

$$\mathbf{A} \text{ is singular} \quad \Leftrightarrow \quad \det \mathbf{A} = 0$$

Algebra of Tensors

Inverse of a Second-order Tensor

For a **non-singular** second-order tensor $\mathbf{A}$, i.e. $\det \mathbf{A} \neq 0$, there exist a unique **non-singular inverse** second-order tensor, denoted as $\mathbf{A}^{-1}$, satisfying,

$$\left(\mathbf{A}\mathbf{A}^{-1}\right)\mathbf{u} = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{u} = \mathbf{1}\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u}$$

yielding,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}, \quad A_{ik} A_{kj}^{-1} = A_{ik}^{-1} A_{kj} = \delta_{ij}$$

Algebra of Tensors

Inverse of a Second-order Tensor

The **inverse** second-order tensor satisfies the following properties,

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T} = \mathbf{A}^{-T}$$

$$\left(\alpha\mathbf{A}\right)^{-1} = \alpha^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1}$$

$$\det\mathbf{A}^{-1} = \left(\det\mathbf{A}\right)^{-1}$$

Algebra of Tensors

Orthogonal Second Order Tensor

An **orthogonal** second-order tensor $\mathbf{Q}$ is a linear operator that preserves the norms and the angles between two vectors,

$$\|\mathbf{Qu}\| = \left(\mathbf{u}\cdot\mathbf{Q}^T\mathbf{Qu}\right)^{1/2} = \left(\mathbf{u}\cdot\mathbf{u}\right)^{1/2} = \|\mathbf{u}\| \quad \forall\mathbf{u}$$

$$\left.\begin{aligned}\mathbf{Qu}\cdot\mathbf{Qv} &= \mathbf{u}\cdot\mathbf{Q}^T\mathbf{Qv} = \mathbf{u}\cdot\mathbf{v}\\ \mathbf{Qv}\cdot\mathbf{Qu} &= \mathbf{v}\cdot\mathbf{Q}^T\mathbf{Qu} = \mathbf{v}\cdot\mathbf{u}\end{aligned}\right\} \quad \forall\mathbf{u},\mathbf{v}$$

and the following relations hold,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{1}, \quad \mathbf{Q}^{-1} = \mathbf{Q}^T, \quad \det\mathbf{Q} = \pm 1$$

Algebra of Tensors

Rotation Second Order Tensor

A **rotation tensor** $\mathbf{R}$ is a proper orthogonal second-order tensor, i.e., is a linear operator that preserves the lengths (norms) and the angles between two vectors,

$$\|\mathbf{Ru}\| = \left(\mathbf{u} \cdot \mathbf{R}^T \mathbf{Ru}\right)^{1/2} = \left(\mathbf{u} \cdot \mathbf{u}\right)^{1/2} = \|\mathbf{u}\| \quad \forall \mathbf{u}$$

$$\left.\begin{aligned} \mathbf{Ru} \cdot \mathbf{Rv} &= \mathbf{u} \cdot \mathbf{R}^T \mathbf{Rv} = \mathbf{u} \cdot \mathbf{v} \\ \mathbf{Rv} \cdot \mathbf{Ru} &= \mathbf{v} \cdot \mathbf{R}^T \mathbf{Ru} = \mathbf{v} \cdot \mathbf{u} \end{aligned}\right\} \quad \forall \mathbf{u}, \mathbf{v}$$

and satisfies the following expressions,

$$\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{1}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = 1$$

Algebra of Tensors

Symmetric/Skew-symmetric Additive Split

A second-order tensor $\mathbf{A}$ can be uniquely additively split into a **symmetric** second-order tensor $\mathbf{S}$ and a **skew-symmetric** second-order tensor $\mathbf{W}$, such that,

$$\mathbf{A} = \mathbf{S} + \mathbf{W}, \quad A_{ij} = S_{ij} + W_{ij}$$

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) = \mathbf{S}^T, \qquad S_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = S_{ji}$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) = -\mathbf{W}^T, \quad W_{ij} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = -W_{ji}$$

Algebra of Tensors

Symmetric/Skew-symmetric Second-order Tensors

The **matrices of components** of a **symmetric** second-order tensor $\mathbf{S}$ and a **skew-symmetric** second-order tensor $\mathbf{W}$, are given by,

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}, \quad [\mathbf{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$$

Algebra of Tensors

Skew-symmetric Second-order Tensors

The **double dot product** of a **symmetric** second-order tensor $\mathbf{S}$ and a **skew-symmetric** second-order tensor $\mathbf{W}$ gives,

$$\begin{aligned}\mathbf{S} : \mathbf{W} &= S_{ij} W_{ij} \\ &= S_{12}W_{12} + S_{13}W_{13} + S_{23}W_{23} - S_{12}W_{12} - S_{13}W_{13} - S_{23}W_{23} \\ &= 0\end{aligned}$$

Algebra of Tensors

Skew-symmetric Second-order Tensor

A **skew-symmetric** second-order tensor $\mathbf{W}$ can be defined by means of an **axial** (or **dual**) **vector** $\boldsymbol{\omega}$, such that,

$$\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u} \quad \forall \mathbf{u}$$

$$\|\mathbf{W}\| = \sqrt{2}\,|\boldsymbol{\omega}|$$

and using index notation the following relations hold,

$$\mathbf{W} = W_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -\varepsilon_{ijk}\omega_k \mathbf{e}_i \otimes \mathbf{e}_j, \quad W_{ij} = -\varepsilon_{ijk}\omega_k$$

$$\boldsymbol{\omega} = \omega_k \mathbf{e}_k = -\frac{1}{2}\varepsilon_{ijk}W_{ij}\mathbf{e}_k, \qquad \omega_k = -\frac{1}{2}\varepsilon_{ijk}W_{ij}$$

Algebra of Tensors

Skew-symmetric Second-order Tensor

Using **matrix notation** and **index notation** the following relations hold,

$$[\mathbf{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$[\boldsymbol{\omega}] = [\omega_1, \quad \omega_2, \quad \omega_3]^T = [-W_{23}, \quad W_{13}, \quad -W_{12}]^T$$

$$\|\mathbf{W}\| = \sqrt{2\left(W_{12}^2 + W_{13}^2 + W_{23}^2\right)} = \sqrt{2}\sqrt{\omega_1^2 + \omega_3^2 + \omega_3^2} = \sqrt{2}\,|\boldsymbol{\omega}|$$

Algebra of Tensors

Spherical Second-order Tensor

A **spherical** second-order tensor $\mathbf{A}$ is defined as,

$$\mathbf{A} = \alpha \mathbf{1}, \quad A_{ij} = \alpha \delta_{ij}$$

and has the following properties,

$$\operatorname{tr} \mathbf{A} = \alpha \operatorname{tr} \mathbf{1} = 3\alpha, \quad A_{ii} = \alpha \delta_{ii} = 3\alpha$$

$$[\mathbf{A}] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Algebra of Tensors

Deviatoric Second-order Tensor

A second-order tensor $\mathbf{A}$ is said to be **deviatoric** if the following condition holds,

$$\operatorname{tr}\mathbf{A} = 0, \quad A_{ii} = 0$$

Algebra of Tensors

Spherical/Deviatoric Additive Split

A second-order tensor $\mathbf{A}$ can be uniquely additively split into a **spherical** second-order tensor $\mathbf{A}^{esf}$ and a **deviatoric** second-order tensor $\operatorname{dev}\mathbf{A}$, such that,

$$\mathbf{A} = \mathbf{A}^{esf} + \operatorname{dev}\mathbf{A}, \qquad A_{ij} = A_{ij}^{esf} + \left(\operatorname{dev} A\right)_{ij}$$

$$\mathbf{A}^{esf} = \frac{1}{3}\left(\operatorname{tr}\mathbf{A}\right)\mathbf{1}, \qquad A_{ij}^{esf} = \frac{1}{3} A_{kk}\delta_{ij}$$

$$\operatorname{dev}\mathbf{A} = \mathbf{A} - \frac{1}{3}\left(\operatorname{tr}\mathbf{A}\right)\mathbf{1}, \quad \left(\operatorname{dev} A\right)_{ij} = A_{ij} - \frac{1}{3} A_{kk}\delta_{ij}$$

Algebra of Tensors

Eigenvalues/Eigenvectors

Given a **symmetric** second-order tensor $\mathbf{A}$, its **eigenvalues** $\lambda_i \in \mathbb{R}$ and associated **eigenvectors** $\mathbf{n}_i$, which define an orthonormal basis along the principal directions, satisfy the following equation (Einstein notation does not applies here),

$$\mathbf{A}\mathbf{n}_i = \lambda_i \mathbf{n}_i, \quad (\mathbf{A} - \lambda_i \mathbf{1})\mathbf{n}_i = \mathbf{0}$$

In order to get a non-trivial solution, the tensor $\mathbf{A} - \lambda_i \mathbf{1}$ has to be singular, i.e., its determinant, defining the **characteristic polynomial**, has to be equal to zero, yielding,

$$p(\lambda_i) = \det(\mathbf{A} - \lambda_i \mathbf{1}) = 0$$

Algebra of Tensors

Eigenvalues/Eigenvectors

Given a **symmetric** second-order tensor $\mathbf{A}$, its **eigenvalues** $\lambda_i \in \mathbb{R}$ and associated **eigenvectors** $\mathbf{n}_i$, which define an orthonormal basis along the principal directions, characterize the **physical nature** of the tensor.

The **eigenvalues** do not depend on the particular orthonormal basis chosen to define the components of the tensor and, therefore, the solution of the **characteristic polynomial** does not depend on the system of reference in which the components of the tensor are given.

Algebra of Tensors

Characteristic Polynomial and Principal Invariants

The **characteristic polynomial** of a second-order tensor $\mathbf{A}$ takes the form,

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{1}) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

where the coefficients of the polynomial are the **principal scalar invariants** of the tensor (invariants in front of an arbitrary rotation of the orthonormal basis) given by,

$$I_1(\mathbf{A}) = \operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2(\mathbf{A}) = \frac{1}{2}\left(\operatorname{tr}^2 \mathbf{A} - \operatorname{tr} \mathbf{A}^2\right) = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$$

$$I_3(\mathbf{A}) = \det \mathbf{A} = \lambda_1 \lambda_2 \lambda_3$$

Algebra of Tensors

Eigenvalues/Eigenvectors

Given a **deviatoric part** of a **symmetric** second-order tensor $\operatorname{dev}\mathbf{A}$, its **eigenvalues** $\lambda_i' \in \mathbb{R}$ and asociated **eigenvectors** $\mathbf{n}_i'$, which define an **orthonormal basis** along the principal directions, satisfy the following equation (Einstein notation does not applies here),

$$\operatorname{dev}\mathbf{A}\mathbf{n}_i' = \lambda_i'\mathbf{n}_i', \quad \left(\operatorname{dev}\mathbf{A} - \lambda_i'\mathbf{1}\right)\mathbf{n}_i' = \mathbf{0}$$

In order to get a non-trivial solution, the tensor $\operatorname{dev}\mathbf{A} - \lambda_i'\mathbf{1}$ has to be **singular**, i.e., its determinant, defining the **characteristic polynomial**, has to be equal to zero, yielding,

$$p\left(\lambda_i'\right) = \det\left(\operatorname{dev}\mathbf{A} - \lambda_i'\mathbf{1}\right) = 0$$

Algebra of Tensors

Characteristic Polynomial and Principal Invariants

The **characteristic polynomial** of the deviatoric part of a second-order tensor $\operatorname{dev} \mathbf{A}$ takes the form,

$$p(\lambda') = \det(\operatorname{dev} \mathbf{A} - \lambda' \mathbf{1}) = -\lambda'^3 + I_1' \lambda'^2 - I_2' \lambda' + I_3' = 0$$

where the coefficients of the polynomial are the **pincipal scalar invariants** of the deviatoric part of the tensor (invariants in front of an arbitrary rotation of the orthonormal basis) given by,

$$I_1'(\operatorname{dev} \mathbf{A}) = \operatorname{tr} \operatorname{dev} \mathbf{A} = 0$$

$$I_2'(\operatorname{dev} \mathbf{A}) = -\frac{1}{2} \operatorname{tr}(\operatorname{dev} \mathbf{A})^2 = -\frac{1}{2}(\lambda_1' \lambda_1' + \lambda_2' \lambda_2' + \lambda_3' \lambda_3')$$

$$I_3(\operatorname{dev} \mathbf{A}) = \det \operatorname{dev} \mathbf{A} = \lambda_1' \lambda_2' \lambda_3'$$

Algebra of Tensors

Eigenvalues/Eigenvectors

The **eigenvalues/eigenvectors** problem for a symmetric second-order tensor $\mathbf{A}$ and its the deviatoric part $\operatorname{dev}\mathbf{A}$ satisfy the following relations (Einstein notation does not applies here),

$$\mathbf{A}\mathbf{n}_i = \lambda_i \mathbf{n}_i, \quad \left(\mathbf{A} - \lambda_i \mathbf{1}\right)\mathbf{n}_i = \mathbf{0}$$

$$\operatorname{dev}\mathbf{A}\mathbf{n}'_i = \lambda'_i \mathbf{n}'_i, \quad \left(\operatorname{dev}\mathbf{A} - \lambda'_i \mathbf{1}\right)\mathbf{n}'_i = \mathbf{0}$$

Then,

$$\left(\operatorname{dev}\mathbf{A} - \lambda'_i \mathbf{1}\right)\mathbf{n}'_i = \left(\mathbf{A} - \left(\lambda'_i + \frac{1}{3}\left(\operatorname{tr}\mathbf{A}\right)\right)\mathbf{1}\right)\mathbf{n}'_i = \left(\mathbf{A} - \lambda_i \mathbf{1}\right)\mathbf{n}_i = \mathbf{0}$$

yielding,

$$\lambda_i = \lambda'_i + \frac{1}{3}\left(\operatorname{tr}\mathbf{A}\right), \quad \mathbf{n}_i = \mathbf{n}'_i$$

Algebra of Tensors

Spectral Decomposition

The **spectral decomposition** of the secod-order unit tensor takes the form,

$$\mathbf{1} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$$

For a second-order unit tensor, all three eigenvalues are equal to one, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and any direction is a principal direction. Then we may take any **orthonormal basis** as principal directions and the **spectral decomposition** can be written,

$$\mathbf{1} = \sum_{i=1,3} \mathbf{n}_i \otimes \mathbf{n}_i = \mathbf{n}_i \otimes \mathbf{n}_i$$

Algebra of Tensors

Spectral Decomposition

The **spectral decomposition** of a symmetric second-order tensor $\mathbf{A}$ takes the form (Einstein notation does not applies here),

$$\mathbf{A} = \sum_{i=1,3} \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i$$

If there are two equal eigenvalues $\lambda = \lambda_1 = \lambda_2$, the **spectral decomposition** takes the form,

$$\mathbf{A} = \lambda \left(\mathbf{1} - \mathbf{n}_3 \otimes \mathbf{n}_3 \right) + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3$$

If the three eigenvalues are equal $\lambda = \lambda_1 = \lambda_2 = \lambda_3$, the **spectral decomposition** takes the form,

$$\mathbf{A} = \lambda \mathbf{1} = \lambda \sum_{i=1,3} \mathbf{n}_i \otimes \mathbf{n}_i = \lambda \mathbf{n}_i \otimes \mathbf{n}_i$$

Third-order Tensors

Third-order Tensors

A **third-order tensor**, denoted as $\boldsymbol{\mathcal{A}}$, may be written as a linear combination of tensor products of three orthonormal basis vectors, denoted as $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$, such that,

$$\boldsymbol{\mathcal{A}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \mathcal{A}_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \mathcal{A}_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

A third-order tensor has $3^3 = 27$ components.

Fourth-order Tensors

Fourth-order Tensors

A **fourth-order tensor**, denoted as $\mathbb{C}$, may be written as a linear combination of tensor products of four orthonormal basis vectors, denoted as $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, such that,

$$\mathbb{C} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \mathcal{C}_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathcal{C}_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

A fourth-order tensor has $3^4 = 81$ components.

Fourth-order Tensors

Double Dot Product with Second-order Tensors

The **double dot product** of a fourth-order tensor $\mathbb{C}$ with a second-order tensor $\mathbf{A}$ is another second-order tensor $\mathbf{B}$ given by,

$$\mathbf{B} = \mathbb{C} : \mathbf{A} = \mathcal{C}_{ijkl} A_{kl} \mathbf{e}_i \otimes \mathbf{e}_j = B_{ij} \mathbf{e}_i \otimes \mathbf{e}_j , \quad B_{ij} = \mathcal{C}_{ijkl} A_{kl}$$

Fourth-order Tensors

Transpose of a Fourth-order Tensor

The **transpose of a fourth-order tensor** $\mathbb{C}$ is uniquely defined as a fourth-order tensor $\mathbb{C}^T$ such that for arbitrary second-order tensors $\mathbf{A}$ and $\mathbf{B}$ the following relationship holds,

$$\mathbf{A} : \mathbb{C} : \mathbf{B} = \mathbf{B} : \mathbb{C}^T : \mathbf{A}, \quad A_{ij} \mathcal{C}_{ijkl} B_{kl} = B_{kl} \mathcal{C}^T_{klij} A_{ij}, \quad \mathcal{C}^T_{ijkl} = \mathcal{C}_{klij}$$

The **transpose of the transpose of a fourth-order tensor** $\mathbb{C}$ is the same fourth-order tensor $\mathbb{C}$,

$$\left(\mathbb{C}^T\right)^T = \mathbb{C}$$

Fourth-order Tensors

Tensor Product of two Second-order Tensors

The **tensor product of two second-order tensors** $\mathbf{A}$ and $\mathbf{B}$ is a fourth order tensor $\mathbb{C}$ given by,

$$\mathbb{C} = \mathbf{A} \otimes \mathbf{B} = A_{ij} B_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathcal{C}_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l ,$$

$$\mathcal{C}_{ijkl} = (\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij} B_{kl}$$

The **transpose of the fourth-order tensor** $\mathbb{C} = \mathbf{A} \otimes \mathbf{B}$ is given by,

$$\mathbb{C}^T = (\mathbf{A} \otimes \mathbf{B})^T = \mathbf{B} \otimes \mathbf{A} = B_{ij} A_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$= \mathcal{C}^T_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l ,$$

$$\mathbb{C}^T_{ijkl} = (\mathbf{A} \otimes \mathbf{B})^T_{ijkl} = (\mathbf{B} \otimes \mathbf{A})_{ijkl} = B_{ij} A_{kl}$$

Fourth-order Tensors

Fourth-order Identity Tensors

Let us consider the following **fourth-order identity tensors,**

$$\mathbb{I} = \delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$\overline{\mathbb{I}} = \delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$\hat{\mathbb{I}} = \frac{1}{2}\left(\mathbb{I} + \overline{\mathbb{I}}\right) = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

Fourth-order Tensors

Fourth-order Identity Tensors

The **fourth-order identity tensor** $\mathbb{I}$ satisfies the following expressions,

$$\mathbb{I} : \mathbf{A} = \delta_{ik}\delta_{jl}A_{kl}\mathbf{e}_i \otimes \mathbf{e}_j = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{A},$$

$$\mathbf{A} : \mathbb{I} = A_{ij}\delta_{ik}\delta_{jl}\mathbf{e}_k \otimes \mathbf{e}_l = A_{kl}\mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{A}$$

The **fourth-order identity tensor** $\mathbb{I}$ is **symmetric**, satisfying,

$$\left.\begin{array}{c} \mathbf{A} : \mathbb{I} : \mathbf{B} = \mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \mathbf{B} : \mathbb{I} : \mathbf{A} \\ \mathbf{A} : \mathbb{I} : \mathbf{B} = \mathbf{B} : \mathbb{I}^T : \mathbf{A} \end{array}\right\} \Rightarrow \quad \mathbb{I} = \mathbb{I}^T$$

Fourth-order Tensors

Fourth-order Identity Tensors

The **fourth-order identity tensor** $\bar{\mathbb{I}}$ satisfies the following expressions,

$$\bar{\mathbb{I}} : \mathbf{A} = \delta_{il}\delta_{jk}A_{kl}\mathbf{e}_i \otimes \mathbf{e}_j = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{A}^T$$

$$\mathbf{A} : \bar{\mathbb{I}} = A_{ij}\delta_{il}\delta_{jk}A_{kl}\mathbf{e}_k \otimes \mathbf{e}_l = A_{lk}\mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{A}^T$$

The **fourth-order identity tensor** $\bar{\mathbb{I}}$ is **symmetric**, satisfying,

$$\left.\begin{aligned} \mathbf{A} : \bar{\mathbb{I}} : \mathbf{B} = \mathbf{A} : \mathbf{B}^T = \mathbf{B} : \mathbf{A}^T = \mathbf{B} : \bar{\mathbb{I}} : \mathbf{A} \\ \mathbf{A} : \bar{\mathbb{I}} : \mathbf{B} = \mathbf{B} : \bar{\mathbb{I}}^T : \mathbf{A} \end{aligned}\right\} \Rightarrow \quad \bar{\mathbb{I}} = \bar{\mathbb{I}}^T$$

Fourth-order Tensors

Fourth-order Identity Tensors

The **fourth-order identity tensor** $\hat{\mathbb{I}}$ satisfies the following expressions,

$$\hat{\mathbb{I}} : \mathbf{A} = \frac{1}{2}\left(\mathbb{I} + \bar{\mathbb{I}}\right) : \mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$$

$$\mathbf{A} : \hat{\mathbb{I}} = \frac{1}{2}\mathbf{A} : \left(\mathbb{I} + \bar{\mathbb{I}}\right) = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$$

The **fourth-order identity tensor** $\hat{\mathbb{I}}$ is **symmetric**, satisfying,

$$\left.\begin{aligned} \mathbf{A} : \hat{\mathbb{I}} : \mathbf{B} = \mathbf{A} : \frac{1}{2}\left(\mathbf{B} + \mathbf{B}^T\right) = \mathbf{B} : \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) = \mathbf{B} : \hat{\mathbb{I}} : \mathbf{A} \\ \mathbf{A} : \hat{\mathbb{I}} : \mathbf{B} = \mathbf{B} : \hat{\mathbb{I}}^T : \mathbf{A} \end{aligned}\right\} \quad \Rightarrow \quad \hat{\mathbb{I}} = \hat{\mathbb{I}}^T$$

Fourth-order Tensors

Fourth-order Deviatoric Projection Operator Tensor

The **fourth-order deviatoric projection operator tensor** $\mathbb{P}_{dev}$ is defined as,

$$\mathbb{P}_{dev} = \mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}, \quad \left(\mathbb{P}_{dev}\right)_{ijkl} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

The **deviatoric part** of a second-order tensor $\mathbf{A}$ can be obtained as,

$$\operatorname{dev}\mathbf{A} = \mathbb{P}_{dev} : \mathbf{A} = \left(\mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right) : \mathbf{A} = \mathbf{A} - \frac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{1},$$

$$\left(\operatorname{dev}\mathbf{A}\right)_{ij} = \left(\mathbb{P}_{dev}\right)_{ijkl} A_{kl} = \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) A_{kl} = A_{ij} - \frac{1}{3}\left(A_{kk}\right)\delta_{ij}$$

Algebra of Tensors

Algebra of Tensors

$$a = \mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad a = u_i v_i$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \varepsilon_{ijk} u_j v_k \mathbf{e}_i, \quad w_i = \varepsilon_{ijk} u_j v_k$$

$$a = \mathbf{u} \cdot \left(\mathbf{v} \times \mathbf{w} \right) = \varepsilon_{ijk} u_i v_j w_k, \quad a = \varepsilon_{ijk} u_i v_j w_k$$

$$\mathbf{A} = \mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j, \quad A_{ij} = u_i v_j$$

$$\mathbf{v} = \mathbf{A}\mathbf{u} = A_{ij} u_j \mathbf{e}_i, \quad v_i = A_{ij} u_j$$

$$\mathbf{C} = \mathbf{A}\mathbf{B} = A_{ik} B_{kj} \mathbf{e}_i \otimes \mathbf{e}_j, \quad C_{ij} = A_{ik} B_{kj}$$

$$a = \mathbf{A} : \mathbf{B} = A_{ij} B_{ij}, \quad a = A_{ij} B_{ij}$$

$$\mathbf{B} = \mathbb{C} : \mathbf{A} = \mathcal{C}_{ijkl} A_{kl} \mathbf{e}_i \otimes \mathbf{e}_j, \quad B_{ij} = \mathcal{C}_{ijkl} A_{kl}$$

$$\mathbb{C} = \mathbf{A} \otimes \mathbf{B} = A_{ij} B_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad \mathcal{C}_{ijkl} = A_{ij} B_{kl}$$

Differential Operators

Nabla

The **nabla** vector differential operator, denoted as ∇, is defined as,

$$\nabla = \frac{\partial}{\partial x_i} \mathbf{e}_i$$

Laplacian

The **laplacian** scalar differential operator, denoted as Δ, is defined as,

$$\Delta = \frac{\partial^2}{\partial x_i^2}$$

Differential Operators

Hessian

The **hessian** symmetric second-order tensor differential operator, denoted as $\nabla \otimes \nabla$, is defined as,

$$\nabla \otimes \nabla = \frac{\partial^2}{\partial x_i \partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$$

Differential Operators

Divergence, Curl and Gradient

The **divergence** differential operator div (·) is defined as,

$$\operatorname{div}(\bullet) = \nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \mathbf{e}_i$$

The **curl** differential operator curl (·) is defined as,

$$\operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_i \times \frac{\partial(\bullet)}{\partial x_i}$$

The **gradient** differential operator grad (·) is defined as,

$$\operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \mathbf{e}_i = \nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i} \mathbf{e}_i$$

Differential Operators

Laplacian

The **laplacian** scalar differential operator can be expressed as the div (grad (·)) operator,

$$\Delta(\bullet) = \nabla \cdot \nabla(\bullet) = \frac{\partial^2}{\partial x_i^2}(\bullet) = \operatorname{div} \operatorname{grad}(\bullet)$$

Hessian

The **hessian** symmetric second order-tensor differential operator can be expressed as the grad (grad (·)) operator,

$$\nabla \otimes \nabla(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i \partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = \operatorname{grad} \operatorname{grad}(\bullet)$$

Differential Operators

Gradient of a Scalar Field

The **gradient differential operator** grad (·) is defined as,

$$\operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \mathbf{e}_i = \nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i}\mathbf{e}_i$$

The **gradient of a scalar field** is a vector field defined as,

$$\boxed{\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i}$$

Differential Operators

Laplacian of a Scalar Field

The **laplacian differential operator** $\Delta(\bullet)$ is defined as,

$$\Delta(\bullet) = \operatorname{div}\operatorname{grad}(\bullet) = \nabla \cdot \nabla(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i^2}$$

The **laplacian of a scalar field** is a scalar field defined as,

$$\Delta\phi = \operatorname{div}\operatorname{grad}\phi = \nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x_i^2} = \phi_{,ii}$$

Differential Operators

Hessian of a Scalar Field

The **hessian differential operator** $\nabla \otimes \nabla(\bullet)$ is defined as,

$$\nabla \otimes \nabla(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i \, \partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$$

The **hessian of a scalar field** is a symmetric second-order tensor field defined as,

$$\nabla \otimes \nabla \phi = \frac{\partial^2 \phi}{\partial x_i \, \partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = \phi_{,ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

Differential Operators

Gradient of a Vector Field

The **gradient differential operator** grad (·) is defined as,

$$\operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \mathbf{e}_i = \nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i}\mathbf{e}_i$$

The **gradient of a vector field** is a second-order tensor field defined as,

$$\operatorname{grad}\mathbf{u} = \nabla \otimes \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = u_{i,j}\mathbf{e}_i \otimes \mathbf{e}_j$$

Differential Operators

Curl of a Vector Field

The **curl differential operator** curl (·) is defined as,

$$\operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_i \times \frac{\partial(\bullet)}{\partial x_i}$$

The **curl of a vector field** is a vector field defined as,

$$\operatorname{curl}\mathbf{u} = \nabla \times \mathbf{u} = \mathbf{e}_j \times \frac{\partial \mathbf{u}}{\partial x_j} = \mathbf{e}_j \times \frac{\partial u_k}{\partial x_j}\mathbf{e}_k = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}\mathbf{e}_i = \varepsilon_{ijk} u_{k,j}\mathbf{e}_i$$

If the **curl** of a **vector field** is **zero**, the vector field is said to be **curl-free** and there is a scalar field such that,

$$\operatorname{curl}\mathbf{u} = 0 \quad \Rightarrow \quad \exists \phi \,|\, \mathbf{u} = \operatorname{grad}\phi$$

Differential Operators

Curl of the Gradient of a Scalar Field

The **curl differential operator** curl (·) and **gradient differential operator** grad (·) are defined, respectively, as,

$$\operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_i \times \frac{\partial(\bullet)}{\partial x_i}, \quad \operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \mathbf{e}_i$$

The **gradient of a scalar field** is a vector field defined as,

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i$$

The **curl of the gradient of a vector field** is a null vector,

$$\boxed{\operatorname{curl}\operatorname{grad}\phi = \nabla \times \nabla\phi = \varepsilon_{ijk}\phi_{,kj}\mathbf{e}_i = \mathbf{0}}$$

Differential Operators

Divergence of a Vector Field

The **divergence differential operator** div (·) is defined as,

$$\operatorname{div}(\bullet) = \nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \mathbf{e}_i$$

The **divergence of a vector field** is a scalar field defined as,

$$\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_j} \cdot \mathbf{e}_j = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial u_i}{\partial x_j} \delta_{ij} = \frac{\partial u_i}{\partial x_i} = u_{i,i}$$

If the **divergence of a vector field** is zero, the vector field is said to be **solenoidal** or **div-free** and there is a vector field such that,

$$\operatorname{div}\mathbf{u} = 0 \quad \Rightarrow \quad \exists\, \mathbf{v} \,|\, \mathbf{u} = \operatorname{curl}\mathbf{v}$$

Differential Operators

Divergence of the Curl of a Vector Field

The **divergence differential operator** div (·) and **curl differential operator** curl (·) are defined, respectively, as,

$$\operatorname{div}(\bullet) = \nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \mathbf{e}_i, \quad \operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_i \times \frac{\partial(\bullet)}{\partial x_i}$$

The **curl of a vector field** is a vector field defined as,

$$\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u} = \mathbf{e}_j \times \frac{\partial \mathbf{u}}{\partial x_j} = \mathbf{e}_j \times \frac{\partial u_k}{\partial x_j} \mathbf{e}_k = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \mathbf{e}_i = \varepsilon_{ijk} u_{k,j} \mathbf{e}_i$$

The **divergence of a curl of a vector field** is a null scalar,

$$\boxed{\operatorname{div} \operatorname{curl} \mathbf{u} = \nabla \cdot (\nabla \times \mathbf{u}) = \varepsilon_{ijk} u_{k,ji} = 0}$$

Differential Operators

Laplacian of a Vector Field

The **laplacian differential operator** $\Delta(\bullet)$ is defined as,

$$\Delta(\bullet) = \nabla \cdot \nabla(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i^2}$$

The **laplacian of a vector field** is a vector field defined as,

$$\Delta \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \frac{\partial^2 u_i}{\partial x_j^2} \mathbf{e}_i = u_{i,jj} \mathbf{e}_i$$

Differential Operators

Divergence of a Second Order Tensor Field

The **divergence differential operator** div (·) is defined as,

$$\operatorname{div}(\bullet) = \nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \mathbf{e}_i$$

The **divergence of a second-order tensor field** is a vector field defined as,

$$\begin{aligned}
\operatorname{div} \mathbf{A} &= \nabla \cdot \mathbf{A} \\
&= \frac{\partial \mathbf{A}}{\partial x_j} \cdot \mathbf{e}_j = \frac{\partial A_{ik}}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_k \cdot \mathbf{e}_j = \frac{\partial A_{ik}}{\partial x_j} \delta_{kj} \mathbf{e}_i \\
&= \frac{\partial A_{ij}}{\partial x_j} \mathbf{e}_i = A_{ij,j} \mathbf{e}_i
\end{aligned}$$

Differential Operators

Curl of a Second-order Tensor Field

The **curl differential operator** curl (·) is defined as,

$$\operatorname{curl}(\bullet) = \nabla \times (\bullet) = \mathbf{e}_i \times \frac{\partial(\bullet)}{\partial x_i}$$

The **curl of a second-order tensor field** is a second-order tensor field defined as,

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \nabla \times \mathbf{A} \\ &= \mathbf{e}_l \times \frac{\partial \mathbf{A}}{\partial x_l} = \mathbf{e}_l \times \frac{\partial A_{kj}}{\partial x_l}\mathbf{e}_k \otimes \mathbf{e}_j \\ &= \varepsilon_{lki}\frac{\partial A_{kj}}{\partial x_l}\mathbf{e}_i \otimes \mathbf{e}_j = \varepsilon_{lki} A_{kj,l}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

Differential Operators

Gradient of a Second-order Tensor Field

The **gradient differential operator** grad (·) is defined as,

$$\operatorname{grad}(\bullet) = \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \mathbf{e}_i = \nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i} \mathbf{e}_i$$

The **gradient of a second-order tensor field** is a third-order tensor field defined as,

$$\begin{aligned}
\operatorname{grad} \mathbf{A} &= \nabla \otimes \mathbf{A} \\
&= \frac{\partial \mathbf{A}}{\partial x_k} \otimes \mathbf{e}_k \\
&= \frac{\partial A_{ij}}{\partial x_k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = A_{ij,k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k
\end{aligned}$$

Differential Operators

Differential Operators

$$\operatorname{grad}\phi = \nabla\phi = \phi_{,i}\mathbf{e}_i, \quad \Delta\phi = \nabla\cdot\nabla\phi = \phi_{,ii}, \quad \nabla\otimes\nabla\phi = \phi_{,ij}\mathbf{e}_i\otimes\mathbf{e}_j$$

$$\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} = u_{i,i}$$

$$\operatorname{grad}\mathbf{u} = \nabla\otimes\mathbf{u} = u_{i,j}\mathbf{e}_i\otimes\mathbf{e}_j,$$

$$\operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u} = \varepsilon_{ijk}u_{k,j}\mathbf{e}_i$$

$$\Delta\mathbf{u} = \nabla\cdot\nabla\mathbf{u} = u_{i,jj}\mathbf{e}_i$$

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = A_{ij,j}\mathbf{e}_i$$

$$\operatorname{grad}\mathbf{A} = \nabla\otimes\mathbf{A} = A_{ij,k}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$$

$$\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} = \varepsilon_{lki}A_{kj,l}\mathbf{e}_i\otimes\mathbf{e}_j$$

Assignments

Assignment 1.3

Establish the following identities involving a smooth scalar field ϕ and a smooth vector field $\mathbf{v}$,

$$(1) \quad \operatorname{div}(\phi \mathbf{v}) = \phi \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \phi$$

$$(2) \quad \operatorname{grad}(\phi \mathbf{v}) = \mathbf{v} \otimes \operatorname{grad} \phi + \phi \operatorname{grad} \mathbf{v}$$

Assignments

Assignment 1.4 [Classwork]

Establish the following identities involving the smooth scalar fields ϕ and ψ, smooth vector fields $\mathbf{u}$ and $\mathbf{v}$, and a smooth second order tensor field $\mathbf{A}$,

$$(1) \quad \operatorname{div}(\phi \mathbf{A}) = \phi \operatorname{div} \mathbf{A} + \mathbf{A} \operatorname{grad} \phi$$

$$(2) \quad \operatorname{div}(\mathbf{A}^T \mathbf{v}) = (\operatorname{div} \mathbf{A}) \cdot \mathbf{v} + \mathbf{A} : \operatorname{grad} \mathbf{v}$$

$$(3) \quad \operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$$

$$(4) \quad \operatorname{div}(\mathbf{u} \otimes \mathbf{v}) = (\operatorname{grad} \mathbf{u}) \mathbf{v} + \mathbf{u} \operatorname{div} \mathbf{v}$$

$$(5) \quad \operatorname{grad}(\phi \psi) = (\operatorname{grad} \phi) \psi + \phi (\operatorname{grad} \psi)$$

Assignments

Assignment 1.5 [Homework]

Establish the following identities involving the smooth scalar field ϕ, and smooth vector fields $\mathbf{u}$ and $\mathbf{v}$,

$$(1)\quad \operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = (\operatorname{grad}^T \mathbf{u})\,\mathbf{v} + (\operatorname{grad}^T \mathbf{v})\,\mathbf{u}$$

$$(2)\quad \operatorname{curl}(\phi\mathbf{v}) = \operatorname{grad}\phi \times \mathbf{v} + \phi \operatorname{curl}\mathbf{v}$$

$$(3)\quad \operatorname{curl}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} + (\operatorname{grad}\mathbf{u})\,\mathbf{v} - (\operatorname{grad}\mathbf{v})\,\mathbf{u}$$

$$(4)\quad \Delta\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v}) - \operatorname{curl}(\operatorname{curl}\mathbf{v})$$

$$(5)\quad \Delta(\mathbf{u}\cdot\mathbf{v}) = (\Delta\mathbf{u})\cdot\mathbf{v} + 2\operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{v} + \mathbf{u}\cdot\Delta\mathbf{v}$$

Assignments

Assignment 1.6 [Classwork]

Given the vector $\mathbf{v} = \mathbf{v}(\mathbf{x}) = x_1 x_2 x_3 \mathbf{e}_1 + x_1 x_2 \mathbf{e}_2 + x_1 \mathbf{e}_3$ determine $\operatorname{div} \mathbf{v}, \operatorname{curl} \mathbf{v}, \operatorname{grad} \mathbf{v}, \Delta \mathbf{v}.$

Integral Theorems

Divergence or Gauss Theorem

Given a vector field $\mathbf{u}$ in a volume V with closed boundary surface ∂V and outward unit normal to the boundary $\mathbf{n}$, the **divergence** (or **Gauss**) **theorem** reads,

$$\int_V \operatorname{div}\mathbf{u}\, dV = \int_V \nabla \cdot \mathbf{u}\; dV = \int_{\partial V} \mathbf{u} \cdot \mathbf{n}\, dS$$

Given a second-order tensor field $\mathbf{A}$ in a volume V with closed boundary surface ∂V and outward unit normal to the boundary $\mathbf{n}$ the **divergence** (or **Gauss**) **theorem** reads,

$$\int_V \operatorname{div}\mathbf{A}\, dV = \int_V \nabla \cdot \mathbf{A}\; dV = \int_{\partial V} \mathbf{A}\mathbf{n}\, dS$$

Integral Theorems

Curl or Stokes Theorem

Given a vector field $\mathbf{u}$ in a surface S with closed boundary ∂S and outward unit normal to the surface $\mathbf{n}$, the **curl** (or **Stokes**) **theorem** reads,

$$\int_S (\operatorname{curl}\mathbf{u})\cdot\mathbf{n}\,dS = \int_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\,dS = \oint_{\partial S}\mathbf{u}\cdot d\mathbf{r}$$

where the curve of the line integral must have positive orientation, such that *dr* points counter-clockwise when the unit normal points to the viewer, following the right-hand rule.