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FUNDAMENTALS OF TRAFFIC FLOW MODELING

CONTENTS

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Introduction to traffic flow modeling

Traffic flow consists in the movement of a large number of vehicles along a particular infrastructure. Although the movement of these individual vehicles may seem random, the overall behavior of the traffic flow is highly predictable. Take as an example the morning traffic accessing big cities, like Barcelona. In normal conditions (i.e. similar demand, no incidents) congestion appears always at same locations, at similar times and evolves in a similar fashion and with similar queue lengths. In this context, it seems reasonable to think that there must exist some physical behavioral laws that steer traffic evolution. The knowledge of these laws would allow to develop traffic flow models, able to predict traffic evolution, congestion and queue lengths. Such models would be very useful in order to assess planning, management and control of road traffic facilities.

This idea of analyzing traffic flow as a science and in quantitative terms appeared in the 1930's with the pioneer works of the first traffic scientist ever, Mr. Bruce Greenshields (https://www.ite.org/about-ite/history/honorary-members/bruce-d-greenshields/). In spite of the early works of Greenshields, it was not until the 1950's that traffic science was popularized, and that the first dynamic models of traffic flow appeared. From the very beginning, two different approaches to flow modelling could be differentiated.

On the one hand, traffic could be seen as a flow, neglecting the fact that it is composed of individual "particles" (i.e. the vehicles). This fluid like approach took researchers to apply hydrodynamic models to traffic, setting the foundations of macroscopic traffic flow modeling. Two physicists in the UK, M.J. Lighthill and G.B. Whitham developed in 1955 the first macroscopic traffic flow model, by comparing "traffic flow on long crowded roads" with "flood movements in long rivers". A year later in the USA, P.I. Richards (1956), physicist and applied mathematician, developed independently the same model, introducing the concept of "shock-waves on the highway" completing the so-called LWR model (i.e. Lighthill-Whitham-Richards model). Since then, this model has received multiple names, like the Continuous Traffic Flow Model, Shock-Wave Theory or Kinematic-Wave Theory. They all refer to the same original and fundamental macroscopic model.

On the other hand, traffic flow could be seen as the aggregated movement of multiple vehicles, so that if individual trajectories are predicted, the overall traffic behavior can be obtained by integration. This trajectory-based approach constitutes the microscopic traffic flow modelling approach. In its most simple form, defined by the unidimensional movement of vehicles in a single lane, models in this category are referred to as car-following models. The concept is that in dense traffic conditions each vehicle follows the trajectory of the vehicle in front, and reacts to its speed changes. By modeling this car-following behavior of vehicles, for all the vehicles in a traffic stream, we could obtain the overall traffic evolution. Seminal car-following models were the ones by L. A. Pipes (1953) based on the California driving code and T.W. Forbes (1958) introducing the concept of "reaction time", both proposing a linear relationship between the distance between two consecutive vehicles (i.e. the spacing) with respect to the traveling speed. Despite these initial works, among the various models of car-following developed over the years, a series of models developed by R.E. Chandler, D.C. Gazis, R. Herman, E.W. Montroll, R.B. Potts and R.W. Rothery at the General Motors (GM) Research Laboratories (Detroit, USA) in the period 1958-61 have received the maximum attention over the years. These car-following models were the first to incorporate the stimulus-response structure, where the stimulus is a differential speed between the leader and follower vehicles and the response of the follower is an acceleration or braking maneuver. It has taken more than half a century to evolve from these first analytical car-following models at the GM research labs to the traffic microsimulators we have today (like this: https://www.youtube.com/watch?v=k_KjM3l295M or this https://www.youtube.com/watch?v=OtYby7QnyAE). Obviously, in addition to a car-following model, we need many other components (e.g. lane-changing model, controllers behavior, O/D matrixes,...) plus an advanced

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digital graphics animation. In spite of being very fancy, you need to know that calibrating these microsimulators requires many parameters (e.g. of the order of 30 maybe), some of them without physical interpretation. This is 10 times more than the 3 parameters we require to calibrate the fundamental diagram in the LWR theory (as you will see next in this lecture). This means that microscopic models are much less robust with respect to LWR, and that all results should be validated with the later.

After this brief introduction to traffic flow modeling and its history, this lecture aims to just present the most fundamental concepts of traffic flow modeling. This includes the definition of traffic variables and the fundamental equation, and the postulates of LWR macroscopic model (i.e. the conservation equation and traffic diagrams). We will end the lecture with an example application of this theory.

Trajectories of a traffic stream on a time-space (x, t) diagram

A trajectory is the representation of the movement of one vehicle in space and time (i.e. x, t). From the trajectory all the details of the movement can be obtained (i.e. position, speed, acceleration) (See Figure 1).

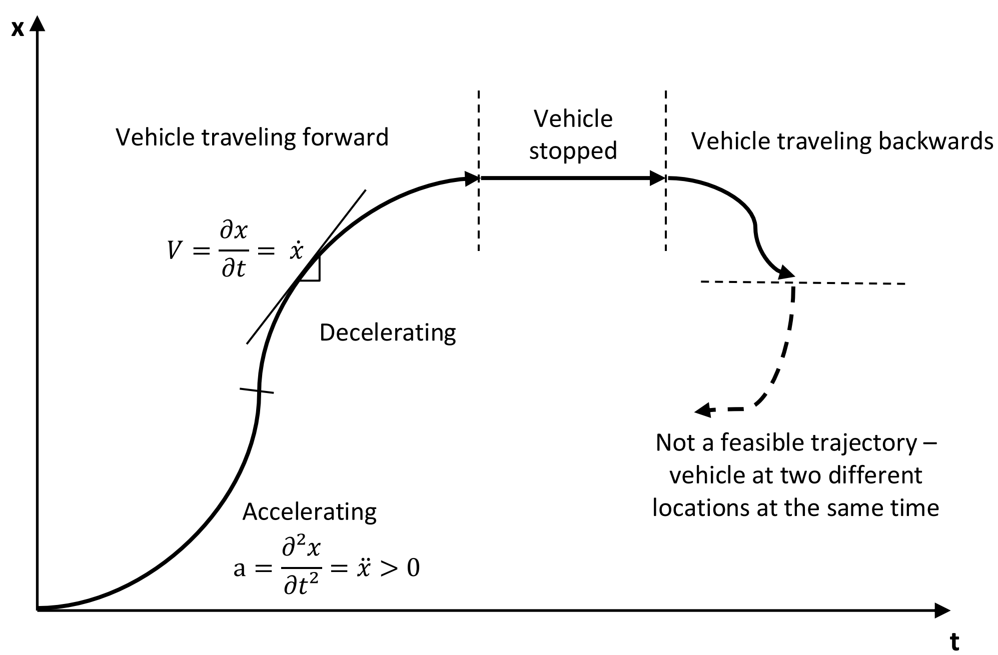

Figure 1. Trajectory definition on a time – space diagram.

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Variables defining a traffic flow: Macroscopic & Microscopic perspective

However, when dealing with traffic flow modeling we are more interested in the variable defining the movement of large amounts of vehicles. Figure 2 and 3 show the trajectories of several consecutive vehicles in a single lane infrastructure, from where we can derive the fundamental variables of traffic flow.

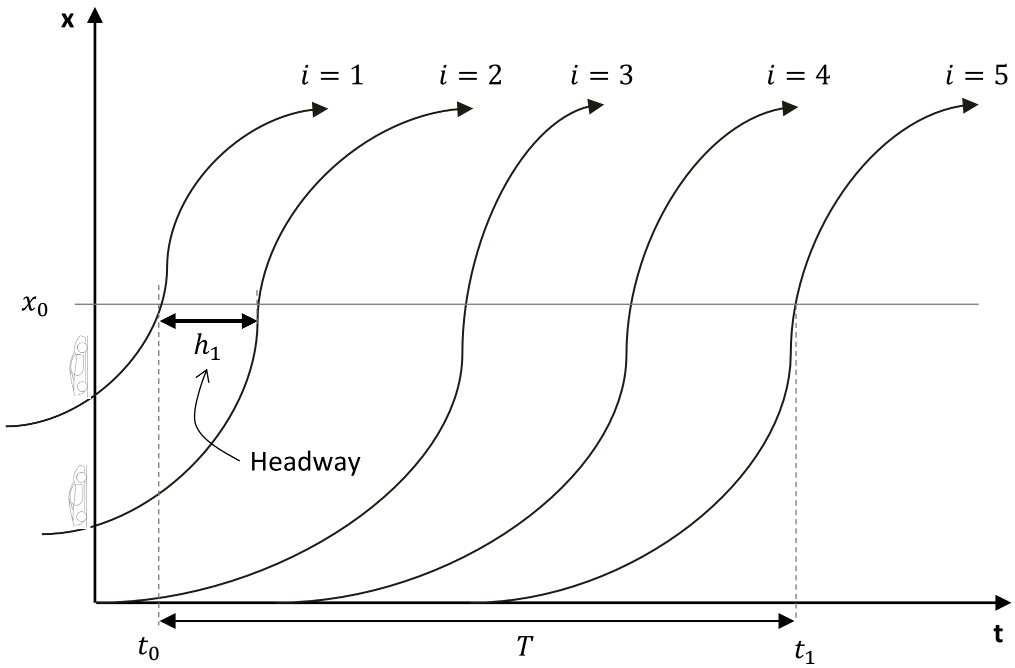

Figure 2. Headway definition on a time – space diagram.

The time interval between the passage of two consecutive vehicles at a given location x_0 is defined as the headway, h, in units of time (e.g. [s]). In Spanish the headway is named "*intérvalo*". Note that the headway is measured taking the same point of consecutive vehicles as the reference (e.g. the rear bumper in Figure 2), so that it includes the time necessary for the passage of the length of the vehicle in addition to the empty time gap between vehicles. Also note that in order to measure the headway, we need to measure at a fixed location continuously in time. This kind of measurement is called a temporal measurement. Consider that we take this temporal measurement during a long period of time, T, and that me measure the passage of m vehicles (e.g. in Figure 2, $m = 5$). Then we can define the traffic flow, q, as:

$$q = \frac{m}{T} \quad [veh/time]$$

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Note that the headway is a traffic variable affecting individual vehicles (i.e. a microscopic traffic variable) while the flow is an aggregated or average variable (i.e. a macroscopic traffic variable). Both variables (i.e. h and q) represent different perspectives (i.e. micro or macro) of the same concept, and they can be related, because:

$$q = \frac{m}{T} = \frac{m}{\sum_{i=1}^{m} h_i} = \frac{1}{1/_m \sum_{i=1}^{m} h_i} = \frac{1}{\bar{h}}$$

So that, the flow is equal to the inverse of the average headway.

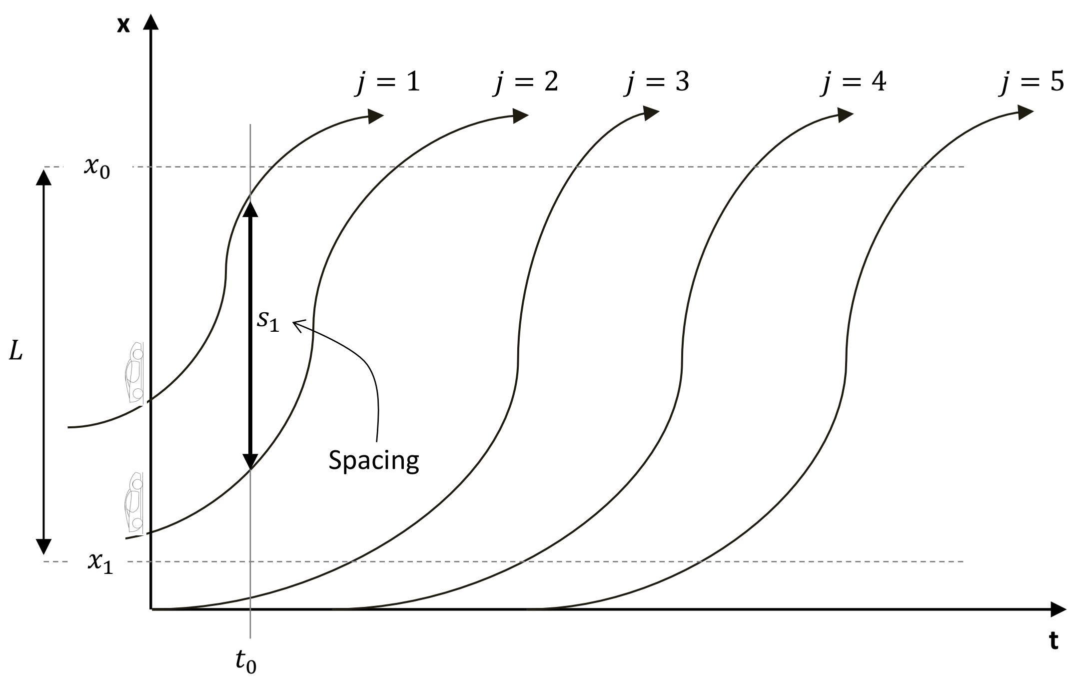

Figure 3. Headway definition on a time – space diagram.

The same analysis could be done but considering a spatial measurement region. This is the simultaneous measurement of all the vehicles in a length L of the infrastructure at a particular instant of time, t_0 (see Figure 3). This allows defining the vehicular spacing, s, as the distance between the same point of consecutive vehicles (e.g. the rear bumper in Figure 3). The spacing, is the second fundamental microscopic traffic variable. Note that, as before, s includes the vehicle length plus the empty space gap between vehicles. In Spanish the spacing is named "*espaciado*". The macroscopic variable equivalent to the spacing is the traffic density k defined as the number of vehicles per unit distance observed at a given instant in a particular infrastructure. The traffic density k can be obtained from the measurement of the existing vehicles, n, on a given infrastructure length, L, at a

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particular instant of time, t_0. Note that in Figure 3, $n = 2$ (i.e. vehicles $j = 1$ and $j = 2$). Then, traffic density is defined as:

$$k = \frac{n}{L} \quad [veh/distance]$$

Similarly, the spacing (microscopic) and the density (macroscopic) variables are related, as one is the aggregation of the other. Namely:

$$k = \frac{n}{L} = \frac{n}{\sum_{j=1}^{n} s_j} = \frac{1}{1/_n \sum_{j=1}^{n} s_j} = \frac{1}{\bar{s}}$$

So that the traffic density can be obtained as the inverse of the average spacing.

The third and last of the fundamental variables is the speed. The microscopic version of the variable is simply the vehicles' individual speed, v_i, (i.e. the slope of the vehicles' trajectory at a given point in time and space). However, the macroscopic average, $\bar{v}$, is a bit more problematic. The average speed is defined as the arithmetic average of the individual speed of vehicles, the problem is that this average depends on which vehicles are considered. Amongst all possible selections of the space-time measurement region, the temporal region (e.g. (x_o, T) and the spatial region (L, t_0), define two particular cases, leading to the time-mean speed, $\bar{v}_t$, and to the space-mean speed, $\bar{v}_s$, respectively (see Figure 4). Time-mean and space-mean definitions, could be applied to any property of the travelling vehicles in addition to the travelling speed.

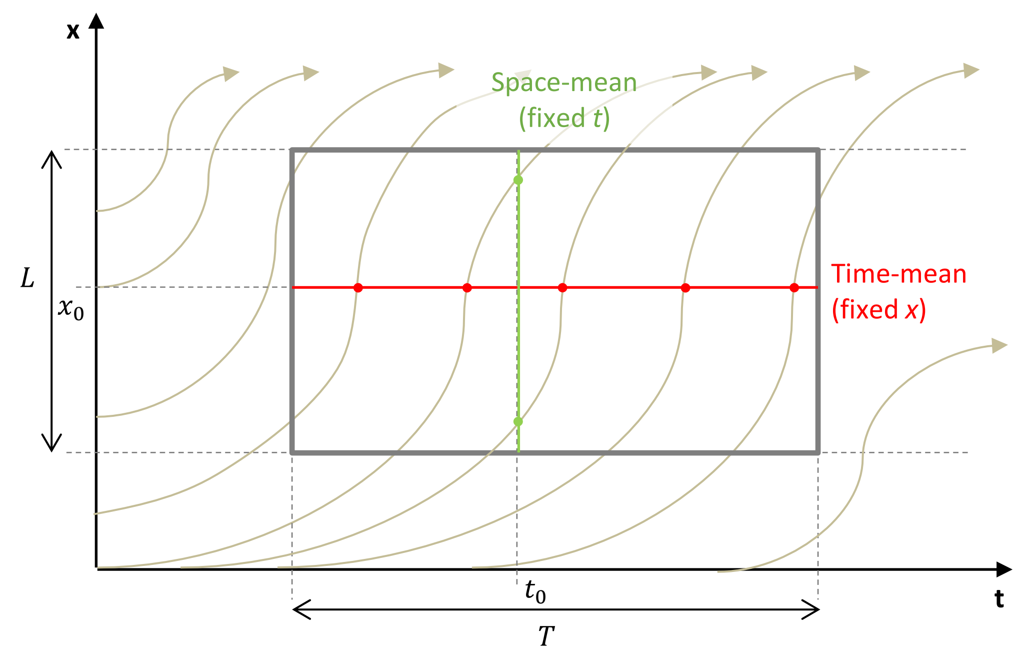

Figure 4. Time vs Space mean averages.

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Then, according to the notation in Figures 2 and 3, we define:

$$\bar{v}_t = \frac{\sum_{i=1}^{m} v_i}{m}$$

$$\bar{v}_s = \frac{\sum_{j=1}^{n} v_j}{n}$$

Because faster vehicles are overrepresented when seen by a stationary observer (i.e. in a temporal measurement region (x_o, T)) for any prevailing traffic state, $\bar{v}_t \geq \bar{v}_s$, and they are equal only when the speeds of all vehicles are constant. More precisely:

$$\bar{v}_t = \bar{v}_s + \frac{\sigma_{\bar{v}_s}^2}{\bar{v}_s}$$

where $\sigma_{\bar{v}_s}^2$ is the variance of the individual speeds measured over the spatial measurement region (L, t_0). Recall that the variance is always positive, and only zero when the variable (i.e. the individual speed, v_i) is constant. The previous relationship is named as the Wardrop relationship, in honor of J.G. Wardrop, the distinguished English mathematician and transport analyst who formulated it the first time in 1952 in his famous publication about "some theoretical aspects of road traffic research".

Note that any traffic state can be thought of being composed of l different families of vehicles (see Figure 5), where within each family traffic is stationary (i.e. constant vehicular speeds, headways and spacings). This is not limiting in any sense, as any non-stationary traffic state can be formulated in these terms (e.g. even a traffic state where all vehicles travel at different speeds, can be thought in terms of "families", where each family is composed of a single vehicle). Then let's define q_l, k_l and v_l, as the flow, density and speed of traffic within family l. Because the flow and density are additive magnitudes, the total flow, q, and the total density, k, can be obtained as:

$$q = \sum_l q_l \, ; \quad k = \sum_l k_l$$

Also, we can state that:

$$\bar{v}_t = \frac{\sum_{i=1}^{m} v_i}{m} = \frac{\sum_l m_l v_l}{m} = \frac{\left(1/_T\right) \sum_l m_l v_l}{\left(1/_T\right) m} = \frac{\sum_l q_l v_l}{q}$$

where m_l is the number of vehicles of family l in the temporal region (x_o, T). This proves that time-means can be obtained as weighted averages, where the weights are the relative flows of each family.

Similarly:

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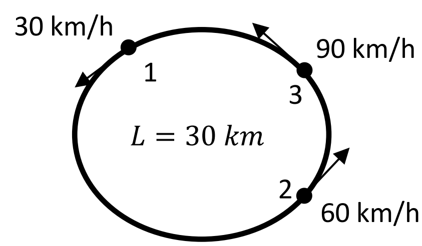
$$\bar{v}_s = \frac{\sum_{j=1}^{n} v_i}{n} = \frac{\sum_l n_l v_l}{n} = \frac{\left(1/_L\right)\sum_l n_l v_l}{\left(1/_L\right) n} = \frac{\sum_l k_l v_l}{k}$$

where n_l is the number of vehicles of family l in the spatial region (L, t_o). This proves that space-means can be obtained as weighted averages, where the weights are the relative densities of each family.

Three vehicles travelling on a circular track at different speeds (i.e. 3 families):

$l = 1 \ \rightarrow v_1 = 90\ km/h$
$l = 2 \ \rightarrow v_2 = 60\ km/h$
$l = 3 \ \rightarrow v_3 = 30\ km/h$

$\bar{v}_t = 70\ km/h$
$\bar{v}_s = 60\ km/h$



$T = 1h$. Every hour:
- 3 vehicles @ 90 km/h ($m_1 = 3$)
- 2 vehicles @ 60 km/h ($m_2 = 2$)
- 1 vehicle @30 km/h ($m_3 = 1$)

$L = 30km$.
- 1 vehicle @ 90 km/h ($n_1 = 1$)
- 1 vehicle @ 60 km/h ($n_2 = 1$)
- 1 vehicle @30 km/h ($n_3 = 1$)

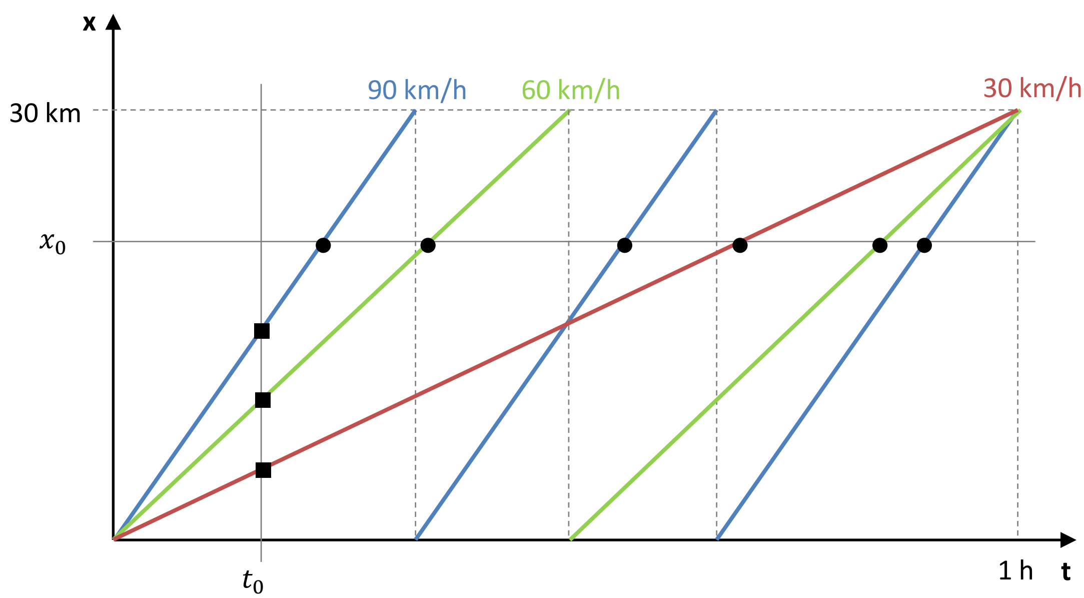


Figure 5. Time vs Space mean speed.

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Table 1 provides a summary of what we have seen so far. Additionally, it includes two additional pieces of information. The "Simulators" column shows computer implementations of the different traffic modeling approaches. The Cell Transmission Model (originally developed by Prof. C.F. Daganzo) is a finite difference implementation of the LWR macroscopic model. This, can be coded into any computing language, and several implementations can be found as freeware. On the other hand, microscopic traffic simulators tend to be commercial software under license, and only SUMO represents an open-source alternative developed by the academic community. Table 1 also provides the macro-micro equivalence for a couple of models. For instance, the Greenberg macroscopic $k - v$ is the macroscopic equivalent of the 3rd generation of the General Motors car-following theories. Take this only as informative, as we are not going to discuss the details of these models in this session.

Table 1. Macro and micro approaches to traffic modelling – Variables, models and simulators

	Variables			Models	Simulators
MACRO	Flow (q)	Density (k)	Average Speed $(\bar{v})$	Continuous theories LWR – KW (shock wave theory)	Cell Transmission Model
MICRO	Headway (h)	Spacing (s)	Instantaneous speed (v_i)	Car following theories	e.g. AIMSUN, VISSIM, PARAMICS, SUMO
MACRO-MICRO Relationships	$q = \frac{1}{\bar{h}}$	$k = \frac{1}{\bar{s}}$	$\bar{v} = \frac{\sum_i^n v_i}{n}$	e.g. Greenberg (macro $k - v$) – 3rd General Motors (micro car following) e.g. Newell Triangular diagram (macro $q - k$)– Forbes min. safety distance (micro car followig)	
Aggregation types	(x, T)	(L, t)	$(x, T) \rightarrow \bar{v}_t$ $(L, t) \rightarrow \bar{v}_s$		

<u>The fundamental equation of traffic ($q = k\bar{v}$)</u>

The three fundamental variables of traffic are related by the so called "Fundamental Equation of Traffic", which states that the flow is equal to the traffic density times the average speed:

$$q = k\bar{v}$$

Obviously, the fundamental equation can also be formulated in terms of the average microscopic variables:

$$\bar{s} = \bar{h}\bar{v}$$

The fundamental equation of traffic results from the previous definitions of the variables, meaning that it is true "by definition". This implies that it holds everywhere, for all kind of infrastructures and for all possible traffic states. In spite of this, for stationary traffic (i.e. constant vehicular speeds, headways and spacings) it is

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particularly easy to prove (see Figure 5). If we carefully define a time-space measurement region (L, T) so that $L/T = \bar{v}$, where $\bar{v} = v_i\ \forall i$ because traffic is stationary, then:

$$q = \frac{m}{T};\ k = \frac{m}{L}$$

$$\frac{q}{k} = \frac{L}{T} = \bar{v}$$

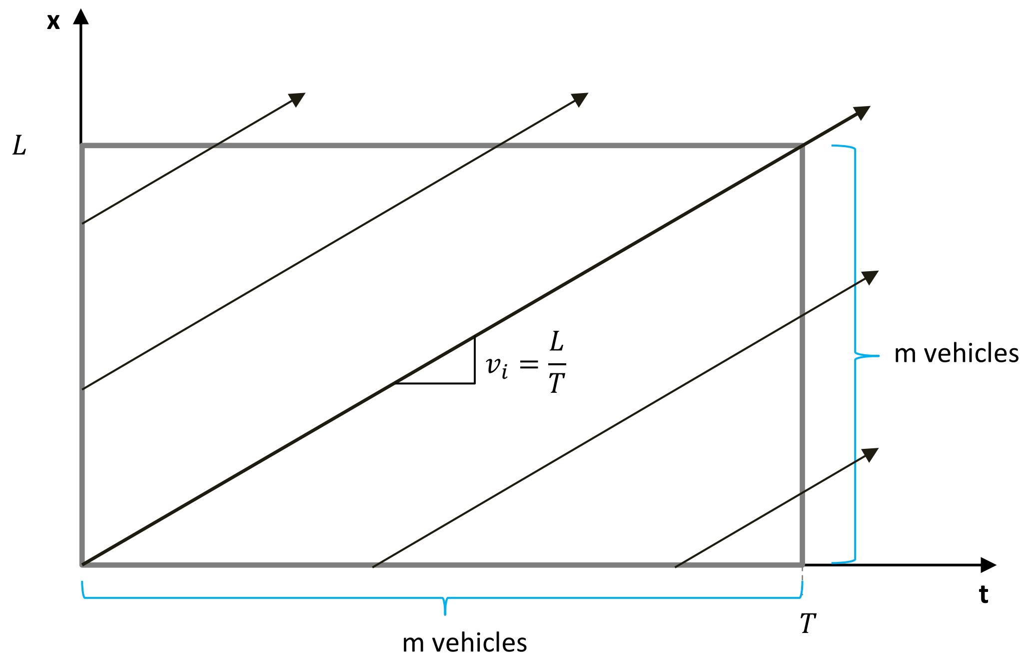

Figure 6. Stationary traffic on a time – space diagram.

In spite of the previous derivation of the fundamental equation of traffic for a stationary traffic state, recall that the equation holds even for non-stationary traffic (i.e. when the vehicular speeds are not constant). Then, an additional issue appears, because the average speed, $\bar{v}$, is not well defined. Should we consider the time-mean speed, $\bar{v}_t$ in the fundamental equation? or should we consider the space-mean speed $\bar{v}_s$? Note that, because they are different in non-stationary traffic, only one of the options can be true.

To give an answer to this question, think of a non-stationary traffic composed of l different families of vehicles. Within each family, traffic is stationary with q_l, k_l, v_l. Note that this is not limiting in any sense, as any non-stationary traffic state can be formulated in these terms (e.g. even a traffic state where all vehicles travel at different speeds, can be thought in terms of "families" composed of a single vehicle). Then, because the flow and density are additive magnitudes, we have that the total flow, q, and the total density, k, are:

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$$q = \sum_l q_l \,; \;\; k = \sum_l k_l$$

And because we have already proved that the fundamental equation holds in stationary traffic, within each family we have:

$$q_l = k_l v_l$$

Then, by simply working with the algebra que can state that:

$$q = \sum_l q_l = \sum_l k_l v_l = k\left[\frac{\sum_l k_l v_l}{k}\right] = k\bar{v}_s$$

Recall that a weighted average of vehicles speeds, where the weights are the relative densities, yields the space-mean speed. So, this proves that in non-stationary traffic, the fundamental equation holds only if the speed is obtained as a space-mean. This is:

$$q = k\bar{v}_s$$

<u>The vehicles' conservation equation</u>

One of the postulates of all models of traffic flow is that vehicles are "conserved". This means that they cannot "disappear", which seems a quite reasonable postulate. The formulation of the vehicles' conservation law is equivalent to any other of the existing conservation laws (mass, energy, etc.) and it states that in any closed system the number of vehicles entering minus the number of vehicles exiting, must be equal to the difference in the number of vehicles stored inside the system.

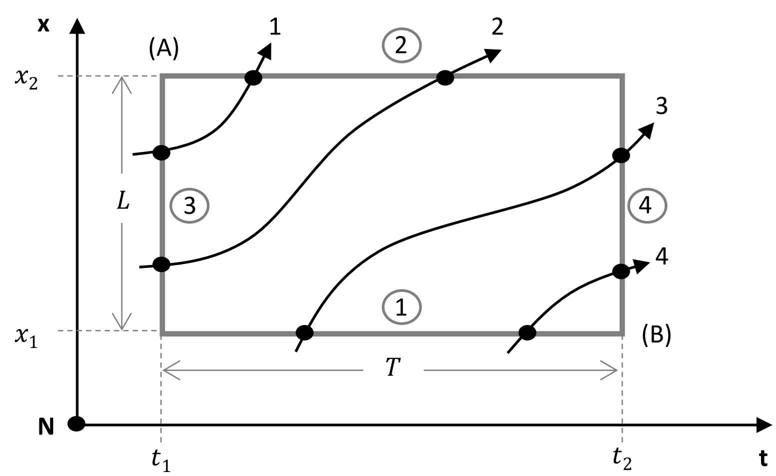

Figure 7. Derivation of the vehicles' conservation equation from a trajectories diagram.

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There are several ways of analytically formulating the vehicles' conservation equation, but using a space-time diagram is quite simple and convenient. In the context of traffic flow, a "closed system" means a closed space-time region, where all the entrances and exits to the infrastructure are monitored. Figure 7 shows a closed (x, t) region with dimensions (L, T) and some vehicle trajectories. The infrastructure considered in Figure 7 could be a section of a one-way street so that vehicles can only enter through location x_1 and exit through x_2. This section of the street is observed in the time period between t_1 and t_2. In this context, note that the vehicles entering the section of the street through x_1 during the period T, (i.e. crossing border 1 of the closed space-time region) are vehicles 3 and 4 (i.e. two vehicles). The notation for this variable will be $m_1 = 2$. The number of vehicles exiting through x_2 during the same period (i.e. crossing border 2) are vehicles 1 and 2. In this case $m_2 = 2$. Also, you can see from Figure 7 that, initially (i.e. at t_1), there were two vehicles inside the section of street of length L. These are vehicles 1 and 2. Because these are spatial borders, the notation used in this case for the number of trajectories crossing border 3 is $n_3 = 2$. Finally, the number of vehicles "stored" inside the section of street at the end of the observation period, t_2 (i.e. trajectories crossing border 4) is $n_4 = 2$ (i.e. vehicles 3 and 4). Given these definitions, the conservation of vehicles can be simply formulated by imposing that the number of trajectories entering the closed (x, t) region must be equal to the number of trajectories exiting. This is:

$$m_1 + n_3 = m_2 + n_4$$

Note that the physical meaning of the previous equation is exactly that of the conservation law:

$$entering(m_1) + initial\ accumulaion(n_3) = exiting(m_2) + final\ accumulation(n_4)$$

And working with the algebra, allows obtaining a more familiar form of the vehicles' conservation equation:

$$m_2 - m_1 = n_3 - n_4$$

$$\frac{m_2 - m_1}{(x_2 - x_1)(t_2 - t_1)} = \frac{n_3 - n_4}{(x_2 - x_1)(t_2 - t_1)}$$

$$\frac{m_1 + n_3}{LT} = \frac{m_2 + n_4}{LT}$$

Note that m's divided by T are flows, while n's divided by L are densities. Therefore:

$$\frac{q_2 - q_1}{(x_2 - x_1)} = \frac{k_3 - k_4}{(t_2 - t_1)}$$

$$\frac{\Delta q}{\Delta x} = -\frac{\Delta k}{\Delta t}$$

Which says that the variation of flow with respect to space is equal to the minus variation of the density with respect to time, representing an equivalent formulation of the conservation equation. Considering that typically

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we deal with many vehicles and that in this case the vehicle count function can be considered continuous, the previous equation can be rewritten as:

$$\frac{\partial q}{\partial x} = -\frac{\partial k}{\partial t}$$

which represents the most typical expression for the vehicles' conservation equation.

Traffic diagrams

The second postulate of macroscopic traffic flow models is the existence of an equation of state. In other words, this means that the state of the system can be univocally determined by applying one equation to a state variable. Typically, this equation of state is plotted in a relevant coordinate axis, defining what is known as a traffic diagram. In the context of traffic flow modelling, there are 3 fundamental variables, which univocally define the traffic state (i.e. q, k, $\bar{v}$). This means that if we want only one degree of freedom (i.e. one state variable), we need two different relationships between these variables. We already know one, the fundamental equation of traffic (i.e. $q = k\bar{v}$). Still, we need another equation between any pair of the variables.

The possibility of obtaining a relationship between a pair of traffic variables appeared from observation. The first traffic observations were performed by the civil engineer B. D. Greenshields, as early as 1933-35. He carried out experiments to measure traffic density and speed using photographic measurement methods for the first time. Measuring traffic with the equipment available at the 1930's, was not an easy task, so that obtaining a single measuring point was a challenging task (see http://www.krbalek.cz/For_students/mds/clanky/Greenshields.pdf) for a more detailed description of the Greenshields experiments). In spite of this, Greenshields managed to obtain 6 measurement points that he represented in a density - speed coordinate axis (see Figure 8). Observing the fairly linear arrangement of measurement points Greenshields proposed a linear relationship between speed and density. This linear k-v model was the first proposal for the equation of state, and the first traffic diagram ever.

From this historical note on the appearance of traffic diagrams, two important properties are already apparent:

- Traffic diagrams are empirical. They are obtained from observation (not from definition as before in the fundamental equation). This means, that they are valid for the infrastructure and the drivers/vehicles observed. So, care must be taken when using traffic diagrams in infrastructures and drivers significantly different from where they were measured.
- Traffic diagrams are obtained from regression to data points. This means that they are true on average, but that it is possible that some measurement point lay significantly apart from the average regression.

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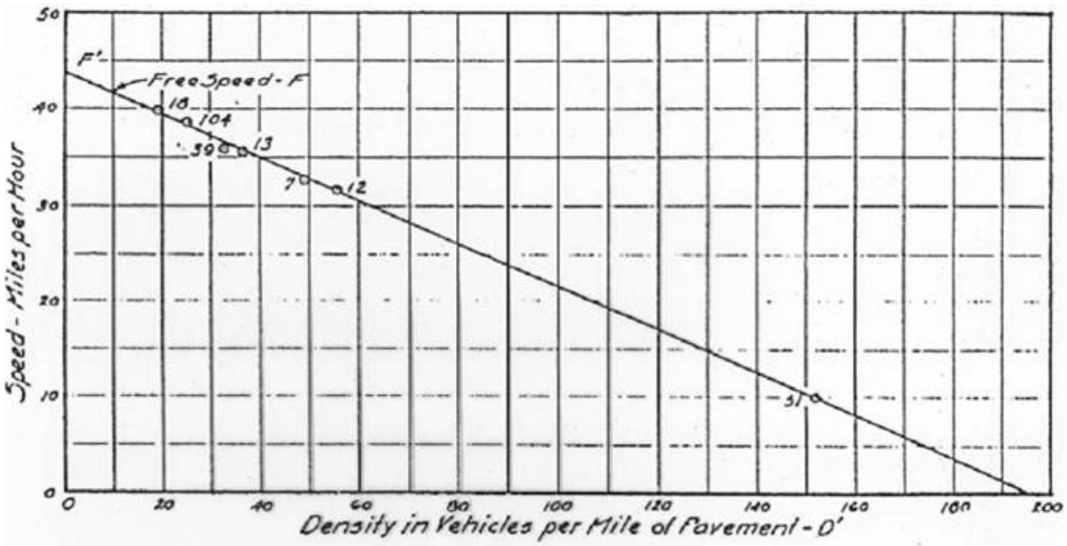

Figure 8. The original Greenshields k-v model

The regression of a functional form to data points for obtaining the traffic diagram (i.e. the calibration of the diagram) yields the parameters of the diagram. The typical parameters of traffic diagrams are:

- v_f, the free flow speed => This is the maximum average speed that drivers feel safe and comfortable to drive at when the density is very low. A typical value for a freeway lane is $v_f = 100 - 120$ km/h, depending on the physical layout, and although this is influenced by the prevailing speed limit.
- k_j, the jam density => This is the maximum density at the infrastructure, when the vehicles are in a gridlock, completely stopped. A typical value for a freeway lane is $k_j = 125 - 150$ veh/km.
- q_{max}, the maximum flow, typically referred as the capacity => This is the maximum throughput that the infrastructure can hold. In turn, v_0 and k_0 represent the corresponding optimal speed and density, respectively, for which the capacity point is obtained. A typical value for a freeway lane is $q_{max} = 2000 - 2200$ veh/h, $k_0 = 20 - 25$ veh/km and $v_0 = 70 - 90$ km/h.

For instance, the Greenshields linear k-v model, requires the calibration of two parameters, v_f and k_j to obtain the linear functional form as:

$$v = v_f \left(1 - \frac{k}{k_j}\right)$$

Figure 9 shows all the previous parameters in the linear Greenshields k-v diagram.

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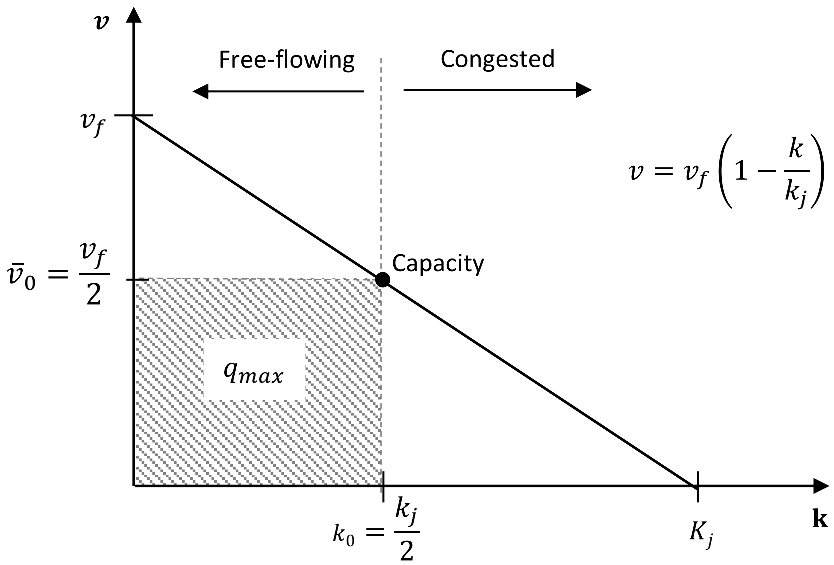

Figure 9. Greenshields $k - v$ linear model (1934).

Note, that the variable not directly represented in the coordinate axis (i.e. the flow in this case) can be obtained by applying the fundamental equation and figuring out the graphical representation on the diagram. In a k-v diagram the flow is obtained as the area defined by the coordinate points of a given traffic state. The capacity point in the Greenshields diagram illustrates that, although this model is of historical importance and academically used (because of its simplicity), it is not accurate. Note for instance that the optimal speed, v_0 results to be $v_f/2$, while in reality should be significantly larger, and $k_0 = k_j/2$, while should be much smaller.

The capacity point divides the diagram in two parts. For densities higher than k_0, the flow is reduced with growing densities. This is the most precise definition of congested traffic. So, the right-hand side of the diagram in Figure 9 corresponds to congested traffic states. In contrast, the left-hand side corresponds to free-flowing traffic states, where an increase of the density is translated to an increase in the flow.

From the functional form of one diagram (e.g. a k-v model) all the other derived diagrams (i.e. all the possible 2-dimensional plots of different combinations of variables) can be obtained using the fundamental equation of traffic. It is a good practice exercise to derive the several different diagrams that would result from the linear Greenshields k-v model. For illustrative purposes, Figure 10 reproduces the conceptual generic form of different possible diagrams.

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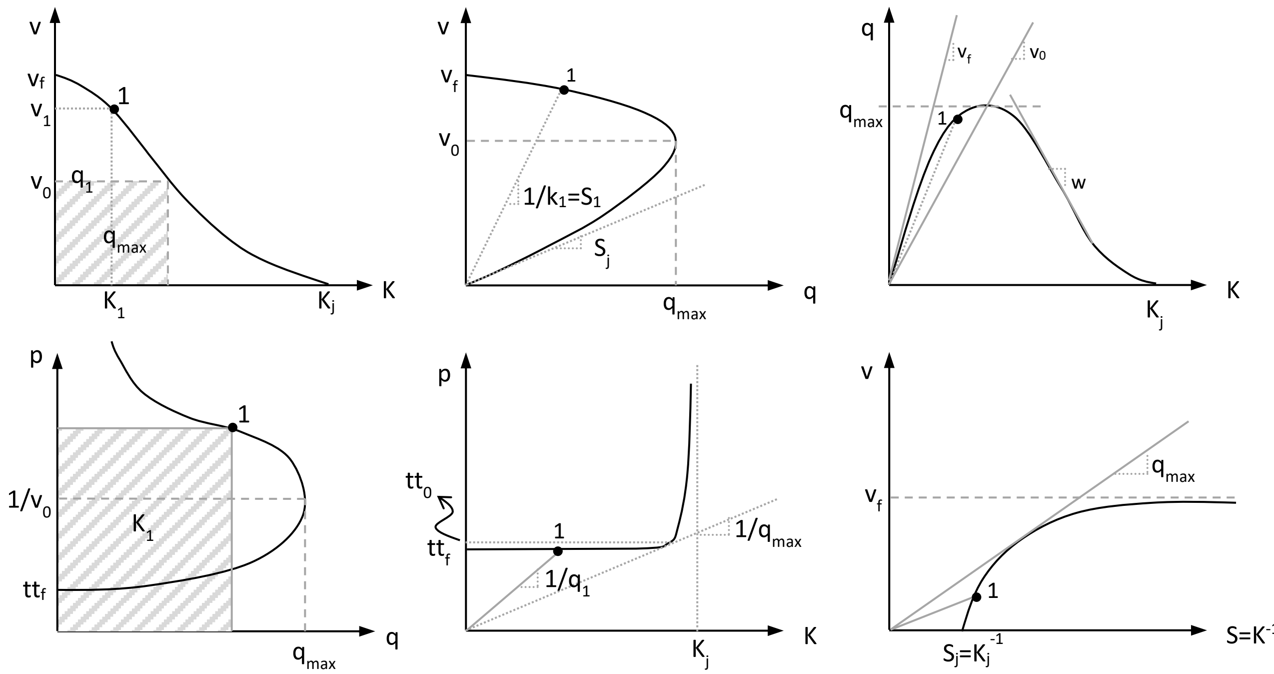

Figure 10. Generic representation of different traffic diagrams
Note: p is defined as the pace, $p = 1/v$, which represents the travel time (tt) per unit distance.

In macroscopic traffic flow modelling, amongst all the possible diagrams, it is of particular importance the diagram relating the flow with the density, (q, k) (top right in Figure 10). This is because it results particularly easy to derive some relevant properties of traffic dynamics by using this graphical representation. This is why this diagram is called the Fundamental Diagram.

Macroscopic modeling of traffic flow LWR - Kinematic Wave Theory

The objective of macroscopic traffic flow modeling is to predict the evolution of traffic variables in time and space. Being macroscopic means that the focus is not on individual vehicles, but on the prediction of the aggregate variables (i.e. q, k and $\bar{v}$). The first and most basic macroscopic approach to traffic flow is the LWR (Lighthill-Whitham-Richards) model, also named the continuous traffic flow theory, the kinematic wave theory, or the shock-wave theory. All refer to the same original model by LWR. The model is based on the hydrodynamic analogy, and its main concept is simple: whenever there is a change in the traffic state a traffic shock-wave is generated. A shock-wave is a transition, an interphase between different traffic conditions (i.e. different q, k or $\bar{v}$). The crossing of a shock-wave informs drivers that they need to adapt to the new traffic conditions. Shock-wave evolve in time and space, they have their own trajectories. It is not the trajectory of any vehicle, but the

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trajectory of information, the trajectory which locates in time and space the changes in traffic conditions. Therefore, if we could predict the evolution of these shock-waves, we could predict the evolution of traffic states and their transitions, precisely the objective of a macroscopic traffic flow model. In summary, LRW theory deals with the prediction of the trajectories of traffic shock-waves. The following link I believe that illustrates quite well the concept of a shock-wave (https://www.youtube.com/watch?v=BtZS7GkPgp4)

The LWR theory assumes two postulates:

- The conservation of vehicles
- The existence of an equation of state (e.g. $q(k)$, the flow as a function of the density, which acts as the state variable). This is the existence of the fundamental diagram.

Speed of a traffic shockwave

LWR theory is based on the prediction of the evolution of shock-waves. So, it fundamental being able to determine the speed of a shock-wave from the knowledge of the "colliding" traffic states. Consider two different traffic states: $U(q^U, k^U)$ upstream and $D(q^D, k^D)$ downstream. In between there needs to exist a transition, a shockwave. If traffic states U and D are stationary (i.e. they do not change neither in time nor in space), the trajectory defined by the shockwave is linear (i.e. constant speed). The objective is to determine the speed, u, of the shockwave given the traffic variables defining U and D (i.e. (q^U, k^U) and (q^D, k^D)).

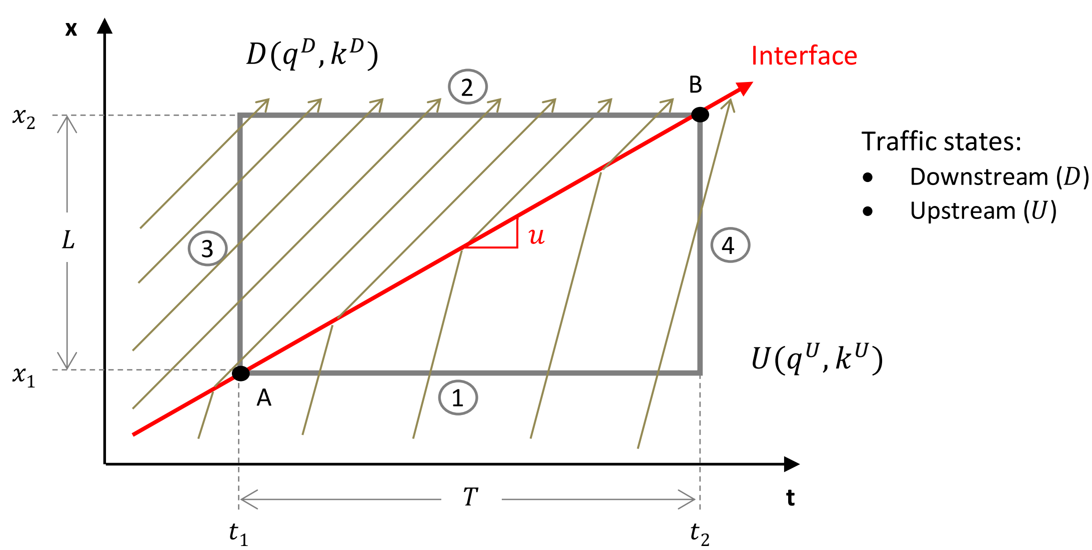

Figure 11. Derivation of the shockwave speed from the conservation equation.

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To that end, the conservation equation can be used. Look at Figure 11 and recall the derivation process for the conservation equation. Then, it should be clear that we can formulate the following:

$$m_1 + n_3 = m_2 + n_4$$

We can rearrange the terms according if they belong to a border of the region in U or D. This is:

$$m_1 - n_4 = m_2 - n_3$$

And then:

$$q^U T - k^U L = q^D T - k^D L$$

$$q^U - k^U \frac{L}{T} = q^D - k^D \frac{L}{T}$$

And note from Figure 11 that by definition $u = L/T$, so that:

$$q^U - k^U u = q^D - k^D u$$

And finally:

$$u = \frac{q^U - q^D}{k^U - k^D} = \frac{\Delta q}{\Delta k}$$

The conclusion is that the speed of the shockwave between two stationary traffic states can be determined by the increase in the flow over the increase in the density. Note that u could be positive (i.e. in the same direction of traffic) or negative (i.e. against traffic), depending on the properties of the colliding traffic states. This solution is of great importance as it allows to predict the evolution of the interphases between different traffic states. Note for instance that if U was a free-flowing traffic state and D a congested one, the interphase between them would be the end of the queue, and the previous expression would allow to predict the evolution of the end of the queue (i.e. the queue extension).

Furthermore, this solution has a straightforward graphical interpretation on the fundamental diagram (see Figure 12), as u, the speed of the shockwave, can be determined by the slope of the line joining the two colliding traffic states.

Then, knowing the initial conditions (i.e. initial traffic states), the contour conditions (i.e. the traffic demand during the analysis period) and the fundamental diagrams for all the sections of the infrastructure under analysis, we can predict the evolution of the shock-waves and the traffic states. The procedure of solving a traffic problem with pen and paper using the LWR model is better illustrated with an example, and with this example we are going to end this lecture.

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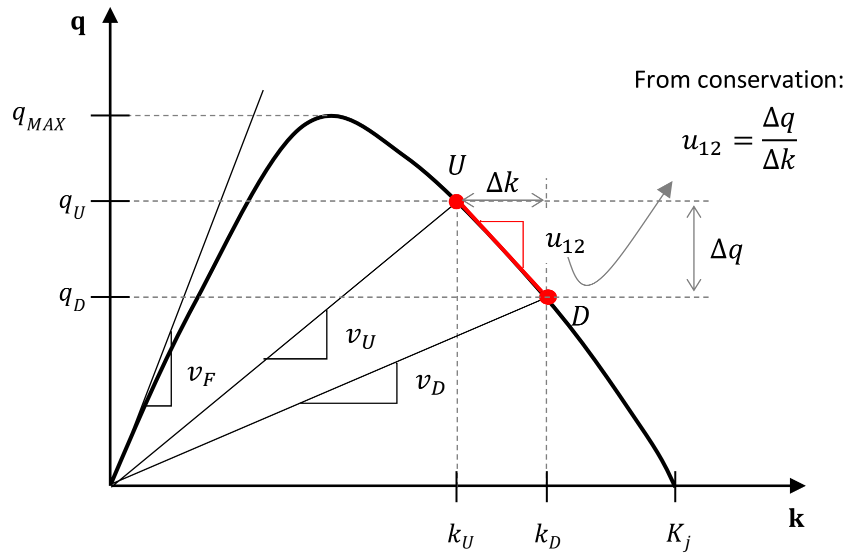
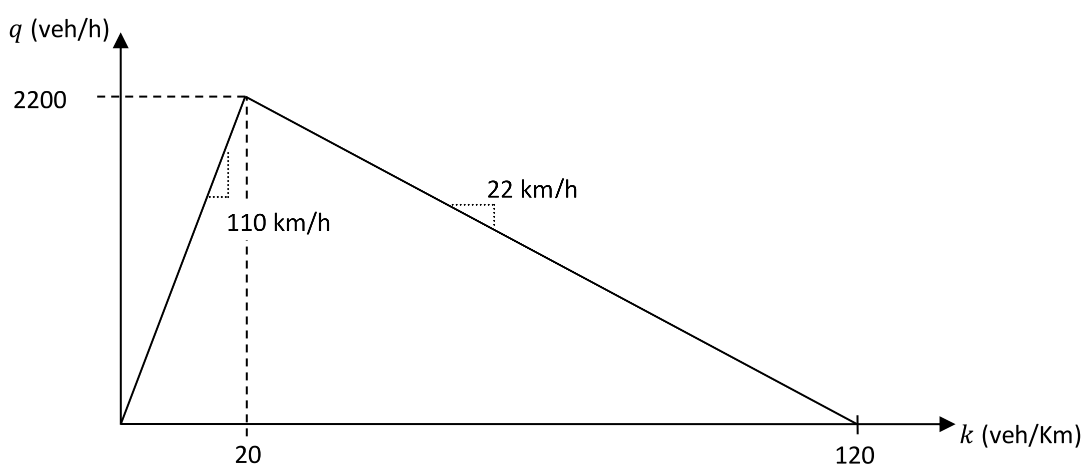


Figure 12. Flow – density diagram: the fundamental diagram

Example of application of the LWR theory: Incident on a freeway

Consider a 6000 veh/h traffic flow travelling on a 3 lane freeway. At time and time $t = 0$ this traffic flow is disrupted by a vehicle breakdown at the location $x = 0$. This incident blocks one lane during 30min. (i.e. from $t = 0$ until $t = 30min$). The figure below represents the flow (q) – density (k) diagram for one lane of the freeway.



And the questions are if this incident generates queues? Why? How far does the queue reach? When does the queue dissipate?

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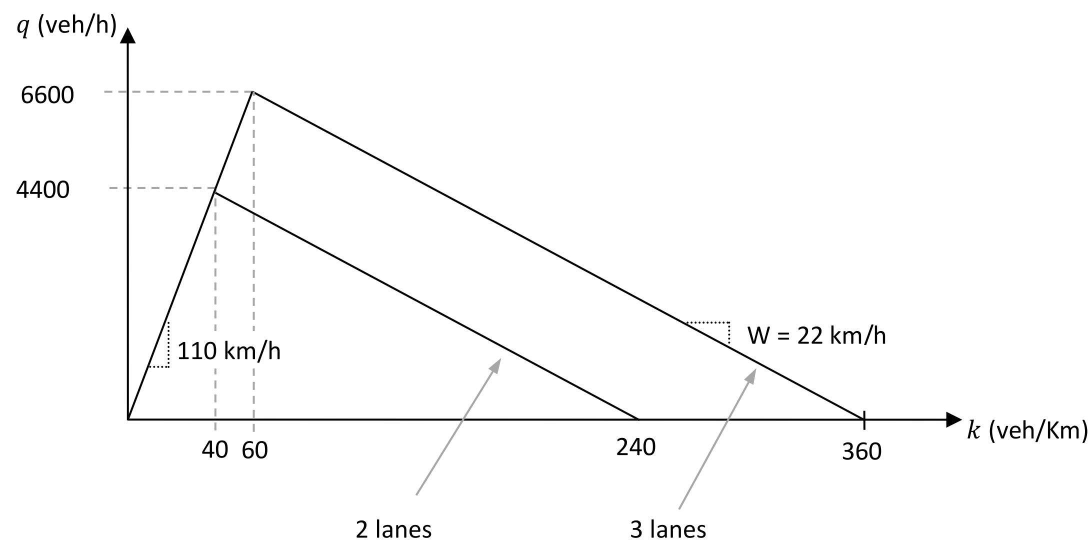
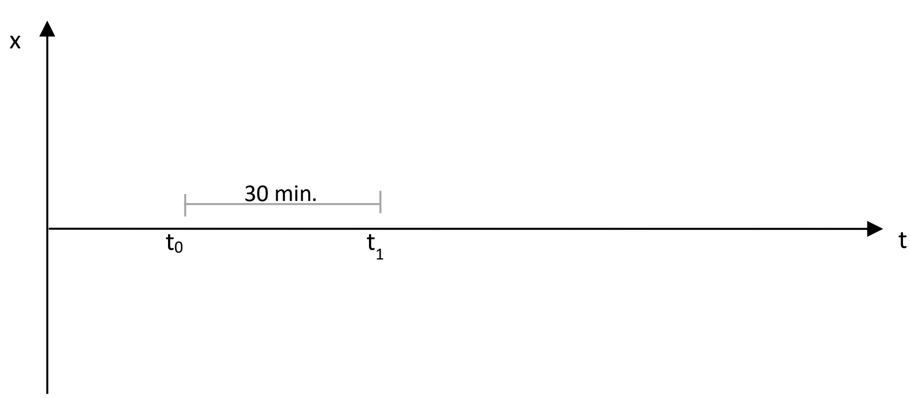
The first thing to do is to obtain the fundamental diagrams for all the sections in the problem. Note that we have a 3-lane freeway, except at the location of the incident that we have 2-lanes. So, we need the diagrams for the 3-lanes sections and for the 2-lane section. From the 1-lane fundamental diagram provided, we can construct the other diagrams assuming that the diagram for a n-lane section exhibits the same speed for n times the density, and we draw the three and the two-lane sections' diagram on the same graph.



Note that the proposed fundamental diagrams are triangular. This is a common simplification in the application of LWR theory, which was initially proposed by prof. G. Newell in the 1990's. He proved that not only simplifies the problem, but also triangular diagrams are in general more accurate that the previously used parabolic shapes.

Next, because the problem needs to predict the evolution of traffic states in (x, t) a space-time diagram needs to be drawn. I recommend locating the $x = 0$ always at a relevant point which allows to visualize upstream and downstream traffic (i.e. somewhere in the middle of the (x, t) diagram.



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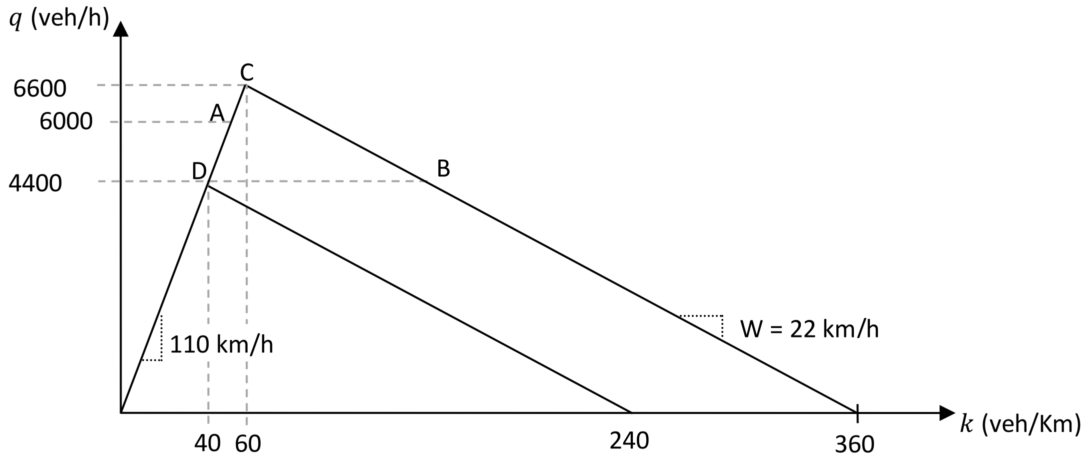
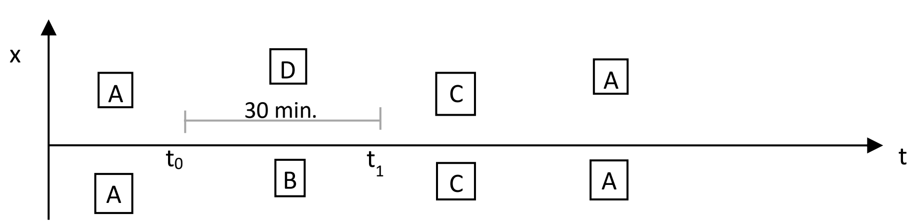
Next step consists in identifying the initial and contour conditions. This is critical, as it will determine the solution. The identification of the intervening traffic states can be done on the fundamental diagrams.



Traffic state A is the traffic demand. This applies before and after the incident for the whole freeway section. Note that the flow of state A is 6000 veh/h, larger than the capacity of 2 lanes, of 4400 veh/h. This implies that the incident will create queues. The queue created at the 2-lane bottleneck will discharge the maximum possible flow (i.e. the capacity of 2-lanes). Therefore, the queue that will appear in the 3-lane freeway upstream of the 2-lane bottleneck will be represented by state B in the previous figure. This is a congested traffic state with the flow equal to 4400 veh/h (i.e. the capacity of 2-lane bottleneck). Downstream of the bottleneck, the flow needs to be the same (i.e. conservation), but traffic will be fre- flowing. This is represented by state D. Finally, one additional traffic state needs to be considered. This is the discharge flow when the full capacity of the 3-lane is recovered (i.e. at $t = 30min$). Queues always discharge at the full available capacity, in this case 6600 veh/h. This is represented by traffic state C.

Once the intervening traffic states have been identified, they can be located on the previous (x, t) diagram.



Finally, it is necessary to identify the shockwaves delimiting each traffic state, which will allow to determine the traffic evolution in the (x, t) diagram. This is done with the help of the fundamental diagram and the graphical derivation of the speeds of the shockwaves.

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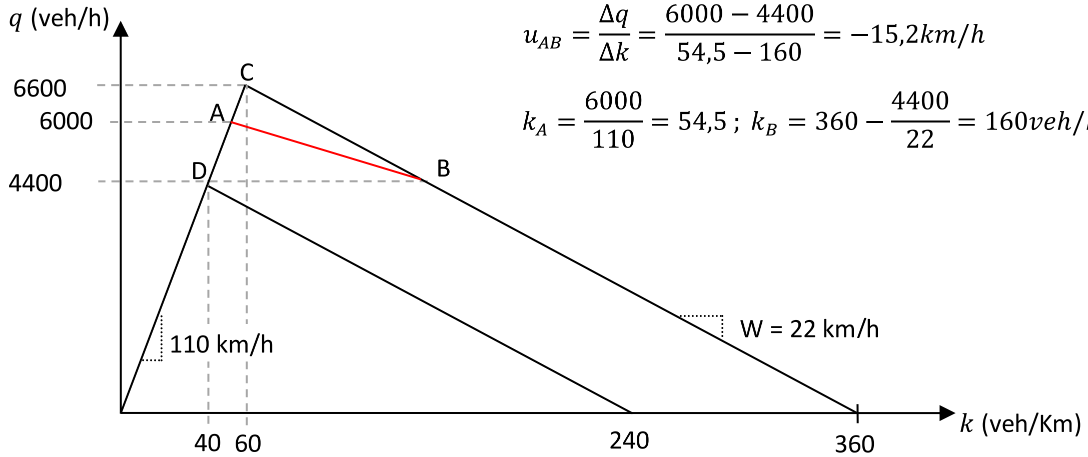
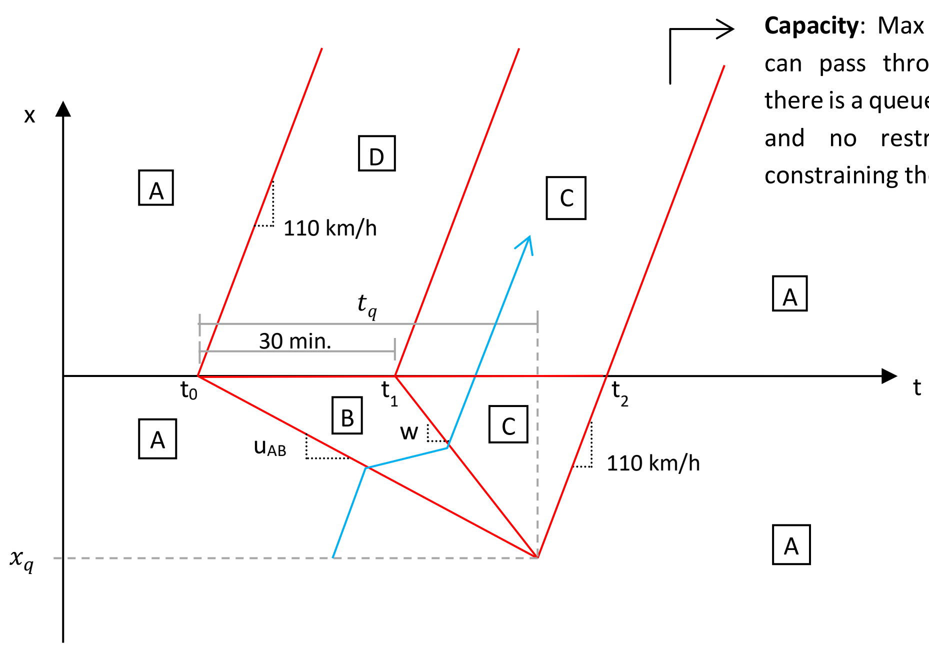


$$u_{AB} = \frac{\Delta q}{\Delta k} = \frac{6000 - 4400}{54{,}5 - 160} = -15{,}2 km/h$$

$$k_A = \frac{6000}{110} = 54{,}5\ ;\ k_B = 360 - \frac{4400}{22} = 160 veh/h$$



Capacity: Max sustained flow that can pass through the link when there is a queue upstream feeding it and no restriction downstream constraining the flow.

And this is the final solution of the problem, as it allows to predict the evolution of the queued state B. Note that the queue grows against traffic at a speed u_{AB} (creation shockwave) and it dissolves when the incident is cleared and also from the incident location and against traffic at a speed $u_{BC} = w$ (dissolving shockwave). When these to waves collide, they determine the end of the queued state B. So, you can observe that the queue has reached the location x_q upstream of the incident and has last a duration t_q since the start of the blockage. Also note that the queue does not discharge completely until t_2, when the effects of the incident at x_0 finally end. If you which,

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you could predict the average trajectory of any vehicle, as you know the prevailing speeds in each traffic state zone. A trajectory in blue is shown as an example.

Below, you can find the necessary calculations to determine these times and locations.

$$u_{AB} \cdot t_q = w \cdot (t_q - 0{,}5)$$
$$15.2 \cdot t_q = 22(t_q - 0{,}5)$$
$$15.2 \cdot t_q - 22 \cdot t_q = -11$$
$$t_q = 1.6h$$

$$x_q = 15.2 \cdot 1.6 = 24.5km$$

$t_0 = 0$ $\qquad$ $t_1 = 0.5h$ $\qquad$ $t_q = 1.6h$ $\qquad$ $t_2 - t = \frac{24.5}{110} = 0.22 \rightarrow t_2 = 1.8h$